In fact if

$$
\begin{aligned}
r_1 &= q_1 r_2 + r_3, \\
r_2 &= q_2 r_3 + r_4, \\
r_3 &= q_3 r_4 + r_5, \\
&\cdots \\
r_{m-2} &= q_{m-2} r_{m-1} + r_m,
\end{aligned} \tag{2}
$$

we pass upwards $q_1$ lines to the top of the column of $r_2$ where the sequence $(r_3, r_2)$ occurs. Taking the column of non-dyads $r_3$ we move upwards $q_2$ lines to the sequence $(r_3, r_4)$ and so on until we at last reach the sequence $(r_{m-1}, r_m)$ or else $(r_m, r_{m-1})$ according as $m$ is even or odd. But $r_1$ and $r_2$, being consecutive terms, are prime to each other so that we at length run into the edge of the table at $(r_{m-1}, 1)$ or $(1, r_{m-1})$.

From the above equations we can write $r_1/r_2$ as a continued fraction:

$$r_1/r_2 = [q_1, q_2, q_3, \cdots, q_{m-2}, r_{m-1}].$$

Since the sequences $(r_{m-1}, 1)$ or $(1, r_{m-1})$ are on the $(r_{m-1}-1)$st line, it follows that $(r_1, r_2)$ occurs on the $n$-th line where

$$n = q_1 + q_2 + q_3 + \cdots r_{m-1} - 1,$$

which is in fact Stern's result 10.

If instead of choosing a sequence $(r_1, r_2)$ we select values of $q_1, q_2, q_3 \cdots r_{m-1}$ such that their sum is $n+1$, and calculate the corresponding continued fraction we will get a sequence $(r_1, r_2)$ on the $n$-th line; for the equations (2) may be solved backwards for the $r_i$ knowing the $q_i$.

Since there are $2^n+1$ terms in the $n$-th line we can form $2^n$ fractions $r_1/r_2$. Since the table is symmetric for every fraction $r_1/r_2$ we have a corresponding fraction $r_2/r_1$. The quotients in the expansion of these two fractions are identical, except that in the case of the proper fraction, the set of quotients is preceded by zero. If we change the complete quotient $1/(r_{m-2})$, which is never unity, to $1/[r_{m-1}-1+(1/1)]$ in the expansion of the proper fraction and disregard the zero quotient $q_0$, the number of quotients and their sum will remain unchanged. Since the sequence $(r_1, r_2)$ occurs but once in the table we have by this device a set of quotients corresponding to each fraction in the line with no two sets identical. The number of these sets is $2^n$ which gives at once the proof of the theorem in the theory of partitions that the number of ways of expressing $n+1$ as the sum of positive integers is $2^n$, sums differing only in the order of their terms being counted as distinct.

If $r_1/r_2$ expands with an odd (even) number of quotients, $r_2/r_1$ has an even (odd) number of quotients. In other words for every expression of $n+1$ as a sum of an odd number of integers, there corresponds one and only one set of even number of integers whose sum is $n+1$. This establishes the theorem that the numbers of ways of expressing $n+1$ as a sum of an odd or even number of integers are the same and are equal to $2^{n-1}$.

By finding all the representations of $n+1$ as the sum of integers we can calculate all the sequences $(r_1, r_2)$ on the $n$-th line. The question now arises how to distribute these sequences on the line. In a particular case the distribution can be effected by tentative methods using the facts 4, 5, 6 and others. But we shall develop a formula which assigns to any number $(r_1)$ in the sequence $(r_1, r_2)$ a definite rank. A given sequence $(r_1, r_2)$ appears on the left or right side of the table according as the number of quotients in the expansion of $r_1/r_2$ is odd or even. We can always suppose that the given sequence $(r_1, r_2)$ is such that the continued fraction for $r_1/r_2$ has an even number of quotients and is such that $r_1 > r_2$.

For if $r_1/r_2$ has an odd number of quotients and is thus on the right side of the table, the fraction $r_2/r_1$ on the left has an even number. If $r_2 > r_1$ then the adjacent sequence $(r_2, r_2 - r_1)$ can be taken in lieu of the given sequence. When we find the rank of $r_1$ on the left side, the rank of $r_1$ on the right can be easily calculated. For example if we wish to find the position of 85 in the sequence (85, 16) we expand the continued fraction $85/16 = [5, 3, 5]$. Thus (85, 16) is on the 12th line on the right hand side. If we know the rank of 85 in (16, 85) which is on the left side of the table we can answer our question by subtracting this rank from $2^{12}+2$. But (16, 85) is followed by (85, 85 − 16) or (85, 69). We find that[1] $85/69 = [1, 4, 3, 5]$ which has in fact an even number of quotients. Referring to equations (2), the rank of a dyad $r_{m-1}$ in the $(r_{m-1}-1)$st line is 2. According to our Theorem 1, the rank of $r_{m-1}$, $q_{m-2}$ lines down is $2^{q_{m-2}}+1$ and that of its right hand neighbour $r_{m-2}$ is $2^{q_{m-2}}$. Then $q_{m-3}$ lines farther down the number $r_{m-2}$ has the rank $2^{q_{m-3}}(2^{q_{m-2}}-1)+1$, and its right neighbour $r_{m-3}$ has the rank

$$2^{q_{m-3}+q_{m-2}} - 2^{q_{m-3}} + 2.$$

Similarily the rank of $r_{m-4}$, $q_{m-4}$ lines farther down is

$$2^{q_{m-4}+q_{m-3}+q_{m-2}} - 2^{q_{m-4}+q_{m-3}} + 2^{q_{m-4}}.$$

Finally the rank of $r_1$ on the right side of the line $n$ is

$$R = 2^{q_1+q_2+q_3+\cdots+q_{m-2}} - 2^{q_1+q_2+\cdots+q_{m-3}} + \cdots - 2^{q_1+q_2} + 2^{q_1}. \tag{3}$$

We shall next consider the problem of finding the number $r_1$ which has a given rank $R$ in a given line $n$. If $R > 2^{n-1}$ we can consider instead the corresponding rank $2^n+2-R$ which is less than $2^{n-1}$. If $R$ is odd, the number $r_1$ is a non-dyad in the $n$-th line and we can follow $r_1$ back to the $(n-k)$th line where it becomes a dyad, and hence has an even rank. By Theorem 1, $k$ is the largest power of 2 in $R-1$ and the rank of $r_1$ in the $(n-k)$th line is $R_1 = (R-1)2^{-k}+1$. Thus we have only to consider the cases in which $R$ is even and less than $2^{n-1}$. Let now $r_2$ be the left neighbour of $r_1$ and let

$$r_1/r_2 = [q_1, q_2\ \ q_3 \cdots q_{n-2}, r_{m-1}]$$

[1] The labor of making the second expansion is obviated by noting that if $r_1/r_2 = [q_1 q_2 q_3 \cdots]$, then $r_1/(r_1-r_2) = [1, q_1-1, q_2 \cdots]$ or $[q_2+1, q_3, q_4 \cdots]$, according as $r_1 > 2r_2$ or $r_1 < 2r_2$.

The rank of $r_1$ is given by equation (3). We have only to determine the quotients $q_i$. The first quotient $q_1$ is simply the highest power of 2 in the given $R$, $q_2$ is the highest power of 2 in $R\cdot 2^{-q_1}-1$, $q_3$ is the highest power of 2 in $(R\cdot 2^{-q_1}-1)2^{-q_2}+1$, and so on.

In this way we can determine the successive partial quotients in the expansion of $r_1/r_2$. Thus far we have not taken into account the number of the line $n$. The complete quotient $r_{m-1}$ is determined as the difference between $n+1$ and the sum of the partial quotients. Thus we see that the complete quotient $r_{m-1}$ is a function of the line $n$ for a given rank $R$. It follows from the theory of continued fractions that the numbers which occupy the same rank in successive lines are in arithmetical progression whose common difference is the numerator of the penultimate convergent. The value of $r_1$ for a given line can now be readily computed from the continued fraction expansion of $r_1/r_2$.

*Example*: What number on the hundredth line has the rank of one million?

$$
\begin{aligned}
2^{q_1}(2^{q_2+q_3+\cdots q_{m-2}} - \cdots + 1) &= 1000000 \qquad q_1 = 6\\
2^{q_2}(2^{q_3+\cdots q_{m-2}} - \cdots - 1) &= 15624 \qquad q_2 = 3\\
2^{q_3}(2^{q_4+\cdots q_{m-2}} - \cdots + 1) &= 1954 \qquad q_3 = 1\\
2^{q_4}(2^{q_5+\cdots q_{m-2}} - \cdots - 1) &= 976 \qquad q_4 = 4\\
2^{q_5}(2^{q_6+\cdots q_{m-2}} - \cdots + 1) &= 62 \qquad q_5 = 1\\
2^{q_6}(2^{q_7+\cdots q_{m-2}} - \cdots - 1) &= 30 \qquad q_6 = 1\\
2^{q_7}(\ \cdots\cdots\cdots + 1) &= 16 \qquad q_7 = 4\\
&\qquad\qquad \textstyle\sum_{i=1}^{7} q_i = 20
\end{aligned}
$$

$$
r_{m-1} = r_8 = 101 - 20 = 81, \qquad r_1/r_2 = [6,3,1,4,1,1,4,81].
$$

Calculating the value of $r_1/r_2$, we have $r_1 = 97139$.

In the preceding discussion we have found answers to our problems in terms of certain algorithms. We shall pass on to the consideration of series of dyads whose values are given by a definite formula.

Theorem 2. *If the sequence $(r_1, r_2)$ occurs in the n-th line with $r_1 > r_2$ and the rank of $r_1$ being $R$, the smallest dyad occuring between the terms $r_1$ and $r_2$ on the line $n+k$ is $r_1+kr_2$ and its rank is $2^kR$.*

Consider the portion of the table in which we are interested.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r_1$ | | | | | | | | $r_2$ |
| $n+1$ | $r_1$ | | | | $r_1+r_2$ | | | | $r_2$ |
| $n+2$ | $r_1$ | | $2r_1+r_2$ | | $r_1+r_2$ | | $r_1+2r_2$ | | $r_2$ |
| $n+3$ | $r_1$ | $3r_1+r_2$ | $2r_1+r_2$ | $3r_1+2r_2$ | $r_1+r_2$ | $2r_1+3r_2$ | $r_1+2r_2$ | $r_1+3r_2$ | $r_2$ |

In the line $n+2$, the smallest dyad between $r_1$ and $r_2$ is $r_1+2r_2$, since $r_1>r_2$.

Applying the same reasoning to the sequence $(r_1+r_2, r_2)$, the smallest dyad on the $(n+3)$rd line is $r_1+3r_2$, etc. The rank of each minimum dyad at each step is one less than that of $r_2$. By Theorem 1, after $k$ moves, the rank of $r_1+kr_2$ is $2^kR$, which is the theorem.

In general if $(r_1, r_1+r_2, r_2)$ be any three consecutive numbers in the $(n+1)$st line, the largest dyad in the $(n+2)$nd line between $r_1$ and $r_2$ is obtained by starting from the dyad $r_1+r_2$ and moving down one line and towards its largest neighbour. Thus if $r_1>r_2$, we would move to the dyad $2r_1+r_2$. Since $r+r_2>r_1$ we would next move down to the right to the dyad $3r_1+2r_2$.

We shall define by a zig-zag move one which is continually descending and changing its direction from left to right and from right to left etc. at each line. A right (left) zig-zag move starts down towards the right (left). Continually applying the above reasoning we have the theorem:

Theorem 3. *If the sequence $(r_1, r_2)$ for $r_1>r_2$ appears in the n-th line the right zig-zag move passes over dyads which are greater than any other elements between $r_1$ and $r_2$ in any line.*

Corollary. *In any line $n>1$ there are two equal terms which are larger than all the other terms on the line.*

This follows at once from considering the sequence (1, 3, 2) and the symmetry of the table. We shall return to these maximum dyads later. We next consider the rank of any dyad after $k$ steps of a right zig-zag move starting with a term of rank $R_0$. The first step brings us down towards the right to $r_1+r_2$, a dyad of rank $R_1=2R_0$ by Theorem 1. The next move takes us down to the left to a dyad of rank $R_2=2R_1-2$, and after $k$ steps we stop on the dyad in question of rank:

$$R_k = 2R_{k-1} - [1 + (-1)^k]. \tag{4}$$

The solution of this difference equation gives us:

$$R_k = \tfrac{1}{3}\{2^k + 2 + [1 + (-1)^{k+1}]\} + 2^k(R_0 - 1).$$

It is easy to verify that this solution satisfies the required recurrence (4) and that for $k=0$ it has the proper value $R_0$. For a left zig-zag move the recurrence is:

$$R_k = 2R_{k-1} - [1 + (-1)^{k+1}],$$

the solution of which is seen to be

$$R_k = \tfrac{1}{3}\{-2^k + 4 - [1 + (-1)^{k+1}]\} + 2^k(R_0 - 1).$$

These zig-zag moves have another important property namely: the dyads passed over are such that any one is equal to the sum of the preceding two dyads. For in the above diagram

$$3r_1 + 2r_2 = (2r_1 + r_2) + (r_1 + r_2) \text{ for the left move,}$$

and

$$2r_1 + 3r_2 = (r_1 + 2r_2) + (r_1 + r_2) \text{ for the right move.}$$

The property is proved by induction.

We have then to deal with sets of dyads which satisfy the difference equation:

$$W_{n+2} = W_{n+1} + W_n.$$

The general theory of the recurring series of the second order has been considered at length by Lucas.[1] The series $W_n$ are determined when one assigns definite values to $W_0$ and $W_1$. If $W_0=0$. $W_1=1$ the series $W_n$ is the celebrated Fibonacci or Pisano series:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \cdots$$

and the value of the $n$-th term is:

$$U_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

The series is fundamental in our discussion. If $W_0$ and $W_1$ assume other values than (0, 1) it can be shown that

$$W_n = W_1U_n + W_0U_{n-1},$$

where $U_n$ is the $n$-th term of the Fibonacci series. Making use of this fact we can write down the value of the dyad occuring at the end of $k$ steps of a right or left zig-zag move. It is only necessary to know the first two dyads $W_0$ and $W_1$.

The maximum dyad on each line deserves special attention. Taking the lines $n=0$, 1 we have for $W_0=1$, $W_1=2$,

$$W_n = 2U_n + U_{n-1} = U_{n+2}.$$

Hence we can state:

Theorem 4: *In any line n the largest dyads have the common value*

$$U_{n+2} = [(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}]/2^{n+2}\sqrt{5};$$

*and their ranks are*

$$R_n = \tfrac{1}{3}(2^r + 2 + [1 + (-1)^{n+1}])$$

*and*

$$2^n + 2 - R_n = \tfrac{1}{3}(2^{n+1} + 4 - [1 + (-1)^{n+1}]).$$

Again if we start with one in the second line and make a right zig-zag move we pass over the dyads 1, 3, 4, 7, 11, 18, $\cdots$. Here $W_0=1$, $W_1=3$. $W_n=3U_n + U_{n-1}$. Substituting the expressions derived for $U_n$ and $U_{n-1}$ we have

$$W_n = 3\left[\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}\right] + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}}{2^{n+1}}.$$

[1] American Journal of Mathematics, vol. 1, pp. 184–240, 289–321.

These numbers are what Lucas terms $V_n$. We can state the following theorem:

Theorem 5: *In any line $n>2$ the largest dyads in the first and last quarters of the line n have the common value;*

$$\frac{(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}}{2^{n+1}}$$

*and the ranks*

$$R=\tfrac{1}{3}\{2^{n-1}+2+[1+(-1)^n]\} \text{ and } \tfrac{1}{3}\{5\cdot 2^{n-1}+4-[1+(-1)^n].$$

Any number of such theorems may be written out.

In general, if the values $W_0$ and $W_1$ are the first two dyads in a right zig-zag move where $W_0$ is in the $n$-th line with the rank $R$, there is on the $(n+k)$th line the dyad

$$(5)\quad W_k=W_1\left(\frac{(1+\sqrt{5})^k-(1-\sqrt{5})^k}{2^k\sqrt{5}}\right)+W_0\left(\frac{(1+\sqrt{5})^{k-1}-(1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}\right)$$

whose rank is

$$R_k=\tfrac{1}{3}\{2^k+2+[1+(-1)^{k+1}]\}+2^k(R_0-1).$$

The same value is found $l$ lines farther down with the rank

$$(6)\qquad R_{k+l}=\frac{2^l}{3}\{2^k(3R_0-2)-1+[1+(-1)^{k+1}]\}+1.$$

Let $m$ be any line with $m>n$. Let $m-n$ be represented as the sum of two integers $k+l$ in all the $m-n$ ways. Then there exist $(m-n+1)$ elements on the $m$-th line whose values are obtained by putting the values of $k$ in (5) and whose ranks are obtained by putting the values of $l$ and $k$ in (6).

It may be remarked that

$$W_k/W_{k-1}=[1,1,1,1,1,\cdots 1,q_{k+1},q_{k+2}\cdots q_{m-2}].$$

It is hopeless to try to account for every dyad in the table by the zig-zag moves, since there exist sequences $(r_1, r_2)$ in the $n$-th line with all possible arrangements of quotients whose sum is $n+1$. We might define other moves to give other types of continued fraction expansions such as 1, 2, 1, 2, 1, 2, $\cdots$ or 1, 2, 3, 1, 2, 3, $\cdots$ and an enormous variety of other simple types. But we would still be at a loss to account for the infinitude of non periodic expansions occuring in the majority of cases.

*Example*: The numbers $r_1=2960276935825111$, $r_2=1679421121698828$ being selected at random, to determine whether the sequence $(r_1, r_2)$ appears in the array and, if so, where. We find that

$$r_1/r_2=[1,1,3,4,1,2,7,1,7,7,1,5,10,7,1,3,1,2,1,10,1,3,1,4,5,9,3,1,6,16].$$

Therefore since $r_1$ and $r_2$ are prime to each other and the sum of the quotients

is 124, the sequence appears in the 123rd line and on the left side of the middle since there is an even number of quotients. In fact the rank is

$$R = 2^{108} - 2^{102} + 2^{101} - 2^{98} + 2^{89} + \cdots + 2^5 - 2^2 + 2$$
$$= 321666940077382478983219549565470.$$

The number of terms on this line is

$$2^{123} + 1 = 10633823966279326983230456482242756609.$$

So the sequence $(r_1, r_2)$ is about 1/33058 of the way across the line. The average value of the terms on this line is

$$(3/2)^{123} = 4562730984784777544048,$$

which is 15 million times larger than the number $r_1$. The largest term on this line is

$$U_{125} = 594251147575126432128751 25$$

whose least rank is

$$R = 3544607988759775661076818827414252204.$$

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## SIMILAR-PERSPECTIVE TRIANGLES

By FRANK EDWIN WOOD, Northwestern University

1. *Introduction.* The properties of projectively defined configurations are often known for special metric cases before the properties of the general case are considered; such however is not always the case, and in particular the metric properties of the configuration arising from two similar-perspective triangles have only been studied since the properties of the general Desargues configuration were obtained. Moreover in the majority of papers upon this subject, the idea of sense has not been considered, so that many of the theorems obtained are either incompletely proved or are incorrectly stated. Various new theorems are stated in this paper, reference being made to any known to be in the literature.

Two triangles will be said to be *similar-perspective* when (a) they are similar, (b) they are in perspective with concurrent lines joining vertices at which are equal angles, (c) all ten points of the Desargues configurations arising from them are finite.[1] In this paper two triangles are regarded as similar-perspective only in case (d) they have the same relative sense, in addition to the other properties.

One readily sees that any point of a Desargues configuration can be taken as the center of two perspective triangles made up from points of the configuration, so that we can speak of *two corresponding triangles*, and of *a vertex (or point) corresponding to a triangle.*

[1] This condition excludes the consideration of two triangles with corresponding sides parallel.

Cominotto[1] seems to have been the first to consider triangles which are at the same time similar and perspective. He ignores the relative sense of the two triangles, and his theorems are incompletely proved.

Speckman[2] has discussed in considerable detail the relations between two similar and perspective triangles with the opposite sense. His papers naturally do not overlap this one, but rather form a complement to it.

Jerabek,[3] however, has considered the field investigated here, and some of his theorems are restated in this paper.

Rohn[4] has considered congruent-perspective triangles, and other figures.

2. *The Desargues configuration arising from two similar-perspective triangles.* Let $A$, $B$, $C$ be arbitrary distinct points not in a straight line, and let $P$ be a point distinct[5] from them, lying upon the circumcircle of the triangle $ABC$. Let $PA$, $PB$, $PC$ meet a second circle through $P$ in the points $A'$, $B'$, $C'$ respectively. Then the triangle $A'B'C'$ is perspective to the triangle $ABC$, $P$ being the center of perspectivity. Moreover, the two triangles are similar, corresponding angles being equal. Let us use the symbol $\backsim$ to mean supplementary, $\angle A \backsimeq \angle B$ meaning that the angle $A$ is equal or supplementary to the angle B. Then we have at once $\angle A \backsimeq \angle CPB \backsimeq \angle C'PB' \backsimeq \angle A'$, etc., and it can be shown in each case (given the order relations upon each of the lines of the configuration) that $\angle A = \angle A'$ etc.

Let $AB$ and $A'B'$ meet at $C''$, $BC$ and $B'C'$ at $A''$, $CA$ and $C'A'$ at $B''$, so that $A''$, $B''$, $C''$ are collinear. Depending upon the order of the points $C''$, $A$, $B$ and of $B''$, $C$, $A$, one has $\angle B''AC'' \backsimeq \angle BAC$, similarly $\angle B''A'C'' \backsimeq \angle B'A'C'$, and since $\angle BAC = \angle B'A'C'$, then $\angle B''A'C'' \backsimeq B''AC''$ so that the four points $B''$, $C''$, $A$, $A'$ lie upon a circle. Similarly one shows that there is a circle through $A''$, $B''$, $C$, $C'$, and a circle through $C''$, $A''$, $B$, $B'$. We also have a circle through $A$, $B$, $C$, $P$ and a circle through $A'$, $B'$, $C'$, $P$. It follows then that each point of the Desargues configuration arising from the triangles $ABC$ and $A'B'C'$ lies upon the circumcircle of both its corresponding triangles, and so is exterior to them, and therefore each pair of corresponding triangles

[1] E. Cominotto, *Una disposizione particolare dei triangoli simili*, Periodico di Matematica, vol. 10 (1894), pp. 103–104; vol. 11 (1895), pp. 59–61.

[2] H. A. W. Speckman, *Eien niewe cirkel in den moderen driehoek*, Nieuw Archief (2) vol. 5 (1902), pp. 367–373; *Over omgekeerd gelijkormige driehoeken, perspectief gelegen* Nieuw Archief (2), vol. 6 (1905), pp. 179–188.

[3] V. Jerabek, *Sur les triangles à la fois semblables et homologiques*, Mathesis (2) vol. 6 (1896), pp. 81–83; *Concerning the theorem of Desargues-Weyr* (written in Bohemian), Casopis, vol. 41, pp. 30–32; this second article was not available to the author.

[4] K. Rohn, *Kongruente Dreiecke, Dreikante, Vierkante und Tetraeder in perspektiver Lage;* Leipzig, Berichte, vol. 71, (1919), pp. 160–192. This article was not available to the author, but in the review of it, in Jahrbuch über die Fortschritte der Mathematik, vol. 47 (1919–20), p. 560, the first theorem attributed to Rohn is incorrect as stated; the paper does not seem to overlap the present article.

[5] If $P$ coincides with one of the three points, say $A$, the line $PA$, thought of as a limiting line as $P$ approaches $A$ along the circumcircle of $ABC$, would be tangent to the circumcircle at $A$. With this understanding, the restriction that $P$ be distinct from $A$, $B$ and $C$ might be removed.

has the same sense. In particular the triangles $ABC$ and $A'B'C'$ have the same sense.

If $A'$, $B'$, $C'$ be the vertices of a triangle perspective to the given triangle, with the center of perspectivity $P$ lying on the circumcircles of $ABC$ and of $A'B'C'$, the configuration just considered would arise, giving

Theorem 1. *If the circumcircles of two perspective triangles meet at the center of perspectivity, the triangles are similar and have the same sense.*

Suppose that $A_1BC$ and $A_1'B'C'$ are perspective triangles with the angles at corresponding vertices equal, and that they have the same sense. Let $P$ be the center of perspectivity. Draw circles circumscribing $CBP$ and $C'B'P$ and let $CA_1$ meet the first circle in $A$, and let $AP$ meet the second circle in $A'$. By theorem 1, the triangles $ABC$ and $A'B'C'$ are similar-perspective. Since $\angle A_1'C'B' = \angle A_1CB = \angle ACB = \angle A'C'B'$, it follows that $A_1'$, $A'$, and $C'$ are collinear. Since the triangles $A_1BC$ and $A_1'B'C'$ are similar, $AC/A'C' = BC/B'C'$, whence $A_1C/A_2'C' = AC/A'C'$. Then either $A_1$ and $A_1'$ coincide with $A$ and $A'$ respectively, or else the lines $AC$ and $A'C'$ are parallel.[1] In the latter case the other corresponding sides of the triangles $A_1BC$ and $A_1'B'C'$ are also parallel. Thus we have proved

Theorem 2. *If two similar triangles with the same sense are so placed that the lines joining corresponding vertices (at which angles are equal) are concurrent, then either the corresponding sides of the two triangles are parallel, or the center of perspectivity is upon the circumcircle of each of the two triangles.*[2]

If the parts of one triangle are arranged in the reverse order to the parts of the other, then Theorem 2 (as others to follow) is not valid.

The triangles $ABC$ and $A'B'C'$ considered in the argument preliminary to Theorem 1 are the most general similar-perspective triangles, from Theorem 2. Using the results obtained in that discussion, we have

Theorem 3. *In the Desargues configuration arising from two similar-perspective triangles, each point of the configuration lies upon the circumcircle of each of its corresponding triangles.*

Theorem 4. *The ten points of a Desargues configuration arising from a pair of similar-perspective triangles lie by fours upon five circles with two circles through each point.*[3]

Let us consider the pair of triangles, $B''CC'$ and $C''BB'$ which are perspective from $A''$; we have noted that the triangles of each pair of corresponding

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[1] Reference theorem: If two given lines $ABC$ and $A'B'C'$ are met by three distinct lines $PAA'$, $PBB'$, $PCC'$ so that $AB:A'B' = BC:B'C'$, then the two given lines are parallel.

[2] This theorem is incorrectly given by Rohn, loc. cit. for a special case.

[3] These two theorems, though not stated by Jerabek (loc. cit.), would follow immediately from relations established by him.

triangles have the same sense, so these triangles $B''CC'$ and $C''BB'$ have the same sense. Moreover the circles circumscribing them pass through their center of perspectivity, $A''$, so they are similar by Theorem 1. A similar argument applies to the other vertices, whence we have

Theorem 5. *If the triangles of one pair of corresponding triangles of a Desargues configuration are similar-perspective, then all ten pairs of corresponding triangles are pairs of similar-perspective triangles.*

Let the triangle $A'B'C'$ vary so as to be similar-perspective to the fixed triangle $ABC$ with a given line $A''B''C''$ as axis. Then the loci of the points $A'$, $B'$, $C'$ are the circles $B''C''A$, $C''A''B$, $A''B''C$, respectively, while the locus of $P$, the center of perspectivity, is the circle $ABC$. Let $A'$ approach $O$, the second intersection of the circles $ABC$ and $B''C''A$, along the latter circle; then $P$ also must approach $O$. Now $B'$ is the intersection of the lines $A'C''$ and $PB$; so when $A'$, and therefore $P$ also, is at $O$, $B'$ is also; likewise $C'$. The point $O$ represents a point triangle similar-perspective to the given triangle $ABC$, and the directions of the three sides are the lines $A''O$, $B''O$, $C''O$, so that the angles between these lines are equal to the angles of the triangle $ABC$. Moreover since $B'$ and $C'$ approach $O$ with $A'$, then the circles $C''A''B$ and $A''B''C$ also pass through $O$. Finally one can show by a similar argument that the circle $A'B'C'$ passes through $O$, whence we have

Theorem 6. *The five circles of theorem* 4 *have a point in common.*[1]

In the preceding paragraph, the triangle $ABC$ and the line $A''B''C''$ were chosen arbitrarily, and form an arbitrary quadrilateral. If each of the four lines in turn be omitted, we get four triangles; so as a special case of the preceding theorem, we have

Theorem 7. *The circumcircles of the four triangles of a quadrilateral formed by omitting one line at a time have a point in common.*[2]

Calling this point the *Miquel point of the triangle and the line* one has, from theorems 4 and 6,

Theorem 8. *In the Desargues configuration arising from two similar-perspective triangles, the twenty Miquel points, one for each triangle of the configuration and the corresponding axis, coincide.*

There are many theorems which might be stated, expressing the relations which have been established; other theorems might be proved as special cases. A few are added.

Theorem 9. *The center of perspectivity of any two similar-perspective triangles is one of the points of intersection of the two circles circumscribing them. The other point is the Miquel point of either triangle and the axis of perspectivity.*

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[1] Jerabek, loc. cit.

[2] Miquel, Liouville's Journal de Mathématiques, vol. 3 (1838), pp. 485–487.

Theorem 10. *Let the vertices of a triangle be joined to a point $P$ upon its circumcircle, and let these joins meet a second circle through $P$ in points of a second triangle; then the second triangle is similar-perspective to the first.*

On the other hand, let $ABC$ and $A'B'C'$ be two similar triangles with the same sense, whose circumcircles meet at a point $P$ upon the line $AA'$; using Theorem 10, one can prove

Theorem 11. *If the join of one pair of corresponding points of two similar triangles with the same sense passes through one of the intersections of their circumcircles, then the joins of the other pairs of corresponding points also pass through the same intersection, and the triangles are similar-perspective.*

By allowing $A'$ to approach $A$, we are led to

Theorem 12. *The tangents to the circles $B''C''A$, $C''A''B$, $A''B''C$ at $A$, $B$, $C$, respectively, meet in a point $Q$ upon the circle $ABC$.*

Now let $\alpha$, $\beta$, $\gamma$, denote the circles $OAA'$, $OBB'$, $OCC'$, respectively; and let $B'$ move along $\beta$ to $A''$; $B'C'$ becomes tangent to $\alpha$ and $A'C'$ becomes tangent to $\beta$, so that the angle between $\alpha$ and $\beta$ is equal or supplementary to the angle $ACB$; and similarly for the angles between the other pairs of circles. Denote the centers of the circles $\alpha$, $\beta$, $\gamma$ by $A_0$, $B_0$, $C_0$, respectively. The line $OA''$ is perpendicular to the line $B_0C_0$; and there is a similar result for the other sides of the triangle $A_0B_0C_0$, which is therefore similar to the triangle $ABC$. Moreover the angle $A_0OC_0$ is equal or supplementary to the angle between the circles $\alpha$ and $\gamma$, and so is equal or supplementary to the angle $A_0B_0C_0$, so that the circle $A_0B_0C_0$ passes through $O$.[1] By repeating the argument with another pair of similar-perspective triangles, one can show that the centers of the circles $ABC$ and $A'B'C'$ lie upon the circle $A_0B_0C_0$, whence we have

Theorem 13. *The centers of the five circles of theorems 4 and 6 lie upon a circle which passes through the point common to all five circles.*

As a special case, we have

Theorem 14. *The centers of the circumcircles of the four triangles of a quadrilateral lie upon a circle which passes through the Miquel point of the quadrilateral.*[2]

3. *Orthohomological triangles.* We have shown that $\angle CB''C' = \angle BC''B'$, $\angle AB''A' = \angle BA''B'$, and $\angle AC''A' = \angle CA''C'$. Now the angle between the lines $AB$ and $A'B'$ may be taken either as the angle $AC''A'$ or as the angle $BC''B'$ in case these two angles are different. By a proper choice of the angle between the lines $AB$ and $A'B'$ when a choice is necessary, and similarly for $BC$ and $B'C'$ and for $CA$ and $C'A'$, there follows

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[1] In this proof, the author follows in part the method of Cominotto, loc. cit.

[2] Gallatly, *The Modern Geometry of the Triangle*, pp. 5–7.

Theorem 15. *The angles between corresponding sides of two similar-perspective triangles are equal.*

When the angles of Theorem 15 are 90°, corresponding sides of the two triangles are perpendicular to each other and the triangles are called orthohomological.[1]

Orthohomological triangles are special cases of similar-perspective triangles and some of the preceding theorems lead to theorems involving orthohomological triangles. If a given pair of similar-perspective triangles are right triangles, then in the Desargues configuration arising from them at least three pairs of corresponding triangles (there are ten pairs in all) are right triangles, and one pair are orthohomological triangles. In a Desargues configuration arising from two orthohomological triangles, at least three pairs of triangles are right triangles.

4. *An application. Theorem of Tafelmacher.*[2] Let $ABC$ be any given triangle and let $ABC_1$ be a triangle with vertices $A$ and $B$ coinciding with the vertices $A$ and $B$ of the triangle $ABC$; let a triangle $AB_1C$ be similarly placed upon the side $AC$ of the triangle $ABC$, the triangle $AB_1C$ being similar to $ABC_1$ with angles at vertices with the same letter, or with a subscript equal; let $A_1BC$ similarly be placed upon $BC$. Let the three triangles $A_1BC$, $AB_1C$, $ABC_1$ have the same sense. Using preceding theorems, especially Theorem 10, one can prove the following

Theorem 16. *If similar triangles $ABC_1$, $AB_1C$, $A_1BC$, all directed inward or all directed outward with respect to a given triangle $ABC$, are placed upon the sides of $ABC$ in such a way that all the angles $A$ are about one point, and likewise for $B$ and $C$, then the lines $AA_1$, $BB_1$, $CC_1$ are concurrent; also the circumcircles of the triangles $A_1BC$, $AB_1C$, $ABC_1$ pass through the point of concurrence.*

The first half of this theorem is due to Tafelmacher.

5. *The nets of conics associated with a Desargues configuration.* The results of section 2 may be extended in part to a general Desargues configuration. Take two conics, one through the points $P$, $A$, $B$, $C$ and the other through the points $P$, $A'$, $B'$, $C'$, of a general Desargues configuration in such a way that the conics have four distinct intersections and so that no one of their three intersections (besides $P$) lies upon a line of the configuration. Let us make a projective transformation taking two of these three points into the circular points at infinity, and the third into any general point of the plane; then the two conics will be transformed into circles, and the Desargues configuration into one arising from two similar-perspective triangles. Therefore we have

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[1] M. J. Neuberg, *Triangles orthohomologiques*, Mathesis (2), vol. 5, (1895), pp. 267–8.

[2] Tafelmacher, *Verallgemeinerung eines von Herrn M. Linnich behandelten Lehrsatzes*, Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, vol. 44 (1913), pp. 315–6. Tafelmacher's proof is analytic.

Theorem 17. *A general Desargues configuration can be projected into some configuration arising from similar-perspective triangles.*

Conversely any general Desargues configuration can be obtained by projection of some configuration arising from similar-perspective triangles; the circles of theorems 4 and 6 project into conics through three points, and so into conics which belong to a net, giving

Theorem 18. *There are five groups of four points each which can be obtained from a Desargues configuration by taking any point of the configuration and the vertices of either of its corresponding triangles. Five conics can be determined each by a group of four points and an arbitrary point of the plane. These five conics belong to a net which has three basis points.*[1]

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## THE NUMBER OF REPRESENTATIONS BY CERTAIN POSITIVE TERNARY QUADRATIC FORMS

By BURTON W. JONES, University of Chicago

1. *Introduction.* The number of solutions of $x^2+y^2+z^2=n$ in integers for any positive integer $n$ is a known function of the class number[2] (a transcendental function) of $n$. More specifically Kronecker[3] proved, in effect: The number of solutions of the equation $x^2+y^2+z^2=n$, where $n$ is neither an exact square nor three times an exact square, is equal to 12 times the difference between the number of reduced non-equivalent forms, $ax^2+2bxy+cy^2$ with $a$ or $c$ odd and the number of such forms with both $a$ and $c$ even, where $a$, $b$ and $c$ are integers, $a>0$ and $ac-b^2=n$. If $n$ is an even square, then to 12 times the difference mentioned we must add 6; if $n$ is an odd square we must subtract 6; if $n$ is 3 times an exact square we add 8.

However, the number of solutions of $ax^2+by^2+cz^2=n$, where $a$, $b$ and $c$ are positive integers, is not, in general, known. The number of solutions of $x^2+2y^2+3z^2=n$, when $n$ is of the form $6m\pm1$, was expressed by Liouville[4] as a function of the class number of $n$, but he gave no proof. Nazimoff[5] proved Liouville's statement for $n$ of the form $6m\pm1$ by methods similar to those below. It should be noted that there is more difficulty when $n$ is a multiple of

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[1] This theorem can be obtained from the work of W. van der Wonde, e.g. *De kubische involutic van den eersten rang in het platte vlak*, Amsterdam Akademie, Verslag, vol. 18 (1910), pp. 842–851.

[2] A set of forms of determinant $n$ with certain restrictions on the coefficients and such that any form of determinant $n$ is equivalent to one of the set is called a set of reduced forms. Thus each reduced form defines a "class" of equivalent forms, the "class number" being the number of such distinct classes. For more detailed explanation see Mathews' *Theory of Numbers*.

[3] Journal für Mathematik, vol. 57 (1860), p. 253.

[4] Journal de Mathématiques, (2), vol. 14 (1869), p. 359.

[5] *Applications of the theory of elliptic functions to the theory of numbers*, translated by Chaimovitch, p. 97.

3. Uspensky[1] has dealt with some such relationships among the above and other forms incidentally, being chiefly concerned with relationships among certain class number functions, and has only very fragmentary results for this problem.

This paper gives the complete solution of the problem: to express the number of solutions of $\beta=n$ in terms of class number functions, where $\beta$ is the form $x^2+2y^2+3z^2$, this being done by giving the number of solutions of $\beta=n$ in terms of the number of solutions of $x^2+y^2+z^2=n'$, where $n'$ is a certain multiple or factor of $n$. Incidentally the same problem is solved for the forms $\alpha=x^2+y^2+2z^2$ and $\gamma=x^2+3y^2+6z^2$.

*The methods used here are applicable not only to these forms but to a wide range of forms including many which, on the surface, seem to have no relation to the form* $\sigma=x^2+y^2+z^2$.[2]

We shall find the following notations useful:

$N(n)$ = the number of integral solutions of $\sigma=n$. $(\sigma=x^2+y^2+z^2)$

$A(n)$ = the number of integral solutions of $\alpha=n$. $(\alpha=x^2+y^2+2z^2)$

$B(n)$ = the number of integral solutions of $\beta=n$. $(\beta=x^2+2y^2+3z^2)$

$C(n)$ = the number of integral solutions of $\gamma=n$. $(\gamma=x^2+3y^2+6z^2)$

$A'(n)$ = the number of integral solutions of $\alpha=n$ with not all three of the variables divisible by 3.

We have the identity:

$$A(9n) = A'(9n) + A(n), \tag{1}$$

since $9n=\xi^2+\eta^2+2\zeta^2$ with $\xi=3\xi'$, $\eta=3\eta'$, $\zeta=3\zeta'$ implies $n=\xi'^2+\eta'^2+2\zeta'^2$ and conversely.

2. *$A(n)$ must first be considered.* If $\sigma$ represents $2n$, there exists a solution $\xi, \eta, \zeta$ such that $\xi^2+\eta^2+\zeta^2=2n$.

If $n\equiv 1 \pmod 2$, one of $\xi, \eta, \zeta$ is even and the other two both odd. From symmetry one third of the solutions $\xi, \eta, \zeta$ of $\sigma=2n$ will thus have $\zeta=2\zeta'$ and $\xi^2+\eta^2\equiv 0 \pmod 2$. Then $\xi+\eta=2\xi'$, $\xi-\eta=2\eta'$ are solvable for $\xi'$ and $\eta'$ and, substituting in $\sigma=2n$, we get $\xi'^2+\eta'^2+2\zeta'^2=n$. Thus, to every solution $(\xi, \eta, 2\zeta')$ of $\sigma=2n$ there corresponds uniquely a solution $(\xi', \eta', \zeta')$ of $\alpha=n$ and conversely, to every solution $(\xi', \eta', \zeta')$ of $\alpha=n$ corresponds uniquely a solution $(\xi, \eta, 2\zeta')$ of $\sigma=2n$. Thus

$$A(n) = \tfrac{1}{3}N(2n) \quad \text{when} \quad n \equiv 1 \pmod 2. \tag{2}$$

If $n\equiv 0 \pmod 2$, $\xi\equiv\eta\equiv\zeta\equiv 0 \pmod 2$. Taking $\zeta=2\zeta'$ and making the above substitution, we have by the same process

$$A(2n) = N(4n). \tag{3}$$

[1] *Bulletin de L'Académie des Sciences de L'Union des Républiques Soviétiques Socialistes*, vols. 19, 20 (1925, 1926): a series of five memoirs. See also the American Journal of Mathematics, vol. 50 (1928), p. 93.

[2] This may be seen from transformations employed in the writer's dissertation, *Representation by positive ternary quadratic forms*, The University of Chicago, June, 1928.

3. $B(n)$ *when* $n \not\equiv 0 \pmod 3$. Consider a solution $\xi, \eta, \zeta$ of $\alpha = 3n$, that is $\xi^2+\eta^2+2\zeta^2 = 3n$. Then either $\xi$ or $\eta \equiv 0 \pmod 3$ but not both. From symmetry, we reduce the number of solutions of $\alpha = 3n$ by one-half if we take $\xi = 3\zeta'$. Then $\eta^2 \equiv \zeta^2 \pmod 3$, $\eta \pm \zeta \equiv 0 \pmod 3$ for one and only one value of the sign and

$$\xi = 3\zeta', \quad \eta \pm \zeta = \pm 3\eta', \quad \xi' = \eta' - \zeta' \tag{4}$$

are solvable for $\zeta', \eta'$ and $\xi'$ uniquely. Substituting in $\alpha = 3n$, we get $\xi'^2+2\eta'^2 +3\zeta'^2 = n$. Now, noting the inverse of the transformation (4)

$$\xi/3 = \zeta', \quad \zeta = \eta' - \xi', \quad \eta = \pm (2\eta' + \xi'), \tag{4'}$$

we see that every solution $(3\zeta', \eta, \zeta)$ of $\alpha = 3n$ determines, by (4), a solution $(\xi', \eta', \zeta')$ of $\beta = n$ which, in turn, determines, by (4'), a pair of solutions $(3\zeta', \pm\eta, \zeta)$ of $\alpha = 3n$. These are distinct since $\eta \not\equiv 0 \pmod 3$. Thus with each pair $(3\zeta', \pm\eta, \zeta)$ of solutions of $\alpha = 3n$ is associated by (4) uniquely one solution $(\xi', \eta', \zeta')$ of $\beta = n$ and conversely. Thus we have

$$B(n) = \frac{1}{4}A(3n) \quad \text{for} \quad n \not\equiv 0 \pmod 3, \tag{5}$$

and, using (2) and (3),

$$\begin{aligned} B(n) &= \frac{1}{4}N(6n) \quad \text{for} \quad n \equiv 2 \quad \text{or} \quad 4 \pmod 6 \\ B(n) &= \frac{1}{12}N(6n) \quad \text{for} \quad n \equiv 1 \quad \text{or} \quad 5 \pmod 6. \end{aligned} \tag{6}$$

4. $B(n)$ *when* $n = 3n'$. As in the previous section, every pair $(3\zeta', \pm\eta, \zeta)$ of solutions of $\alpha = 3n$ with $\eta\zeta \not\equiv 0 \pmod 3$ is associated by (4) with a solution $(\xi', \eta', \zeta')$ of $\beta = n$ with $\xi' \not\equiv \eta' \pmod 3$, since $\xi' = \eta' - \zeta$, and therefore with $\xi' \equiv -\eta' \not\equiv 0 \pmod 3$ since $\xi'^2+2\eta'^2+3\zeta'^2 = 3n'$ and conversely to every solution $(\xi', \eta', \zeta')$ of $\beta = n$ with $\xi' \equiv -\eta' \not\equiv 0 \pmod 3$ corresponds uniquely a pair of solutions $(3\zeta', \pm\eta, \zeta)$ of $\alpha = 3n$ with $\eta\zeta \not\equiv 0 \pmod 3$. On the other hand, to every pair of solutions $(\xi', \eta'', \zeta')$, $(\xi', -\eta'', \zeta')$ of $\beta = n$ with $\xi'\eta'' \not\equiv 0 \pmod 3$ corresponds one solution with $\xi' \equiv -\eta' \not\equiv 0 \pmod 3$ and which, by (4), determines uniquely a pair of solutions $(3\zeta', \pm\eta, \zeta)$ of $\alpha = 3n$ with $\eta\zeta \not\equiv 0 \pmod 3$. Thus, the number of solutions of $\alpha = 3n$ with $\eta\zeta \not\equiv 0 \pmod 3$ and $\xi \equiv 0 \pmod 3$ is equal to the number of solutions of $\beta = n$ with $\xi'\eta' \not\equiv 0 \pmod 3$.

Furthermore, if $\eta\zeta \equiv 0 \pmod 3$, $\eta \equiv \zeta \equiv 0 \pmod 3$, since $\xi = 3\zeta'$, and we have a representation of $9n'$ by $\alpha$ with the variables all divisible by 3. Thus, the number of representations of $3n$ by $\alpha$ with $\eta\zeta \not\equiv 0 \pmod 3$ is $\frac{1}{2}A'(9n')$. Also, if $\xi'\eta' \equiv 0 \pmod 3$, $\xi' \equiv \eta' \equiv 0 \pmod 3$ since $\beta = 3n'$. Suppose we have a solution $\eta' = 3\eta''$, $\xi' = \xi''$, $\zeta'$ of $\beta = n$. Then, substituting in $\beta = 3n'$, we have $3\xi''^2+6\eta''^2 +\zeta'^2 = n'$ and thus for every solution of $\beta = n$ with $\xi'$ and $\eta'$ both divisible by 3 there is a unique solution of $\gamma = n'$ and conversely. Thus the number of representations of $\beta = n$ with $\xi'\eta' \not\equiv 0 \pmod 3$ is $B(3n') - C(n')$ and we have therefore

$$B(3n') - C(n') = \tfrac{1}{2}A'(9n') = \tfrac{1}{2}A(9n') - \tfrac{1}{2}A(n'). \tag{7}$$

If $n' \equiv 2 \pmod 3$, $C(n') = 0$ and we have

$$B(3n') = \tfrac{1}{2}A(9n') - \tfrac{1}{2}A(n'), \quad n' \equiv 2 \pmod 3. \tag{8}$$

If $n' = 3n''$, $\gamma = n'$ implies $z \equiv 0 \pmod 3$ and thus $C(n') = B(n'')$ and we have the reduction formula:

$$B(9n'') = (Bn'') + \tfrac{1}{2}A(27n'') - \tfrac{1}{2}A(3n''). \tag{9}$$

For the consideration of the remaining case: $n' \equiv 1 \pmod 3$ we need

5. *$C(n')$ where $n' \equiv 1$ (mod 3).* Suppose $\xi^2 + \eta^2 + 2\zeta^2 = n'$ with $\xi\eta\zeta \not\equiv 0 \pmod 3$. Then $\eta^2 \equiv \zeta^2 \pmod 3$, $\zeta \pm \eta \equiv 0 \pmod 3$ holds for one and only one of the signs and

$$\eta \pm \zeta = \pm 3\eta', \quad \zeta' = \eta' - \zeta \tag{10}$$

are solvable for $\eta'$ and $\zeta'$ with the proper choice of sign. Substituting $\eta = \pm(3\eta' - \zeta)$ in $\alpha = n'$, we get $\xi^2 + 6\eta'^2 + 3\zeta'^2 = n'$. Noting the inverse transformation we see, as in §3, that with every pair of solutions $(\xi, \pm\eta, \zeta)$ of $\alpha = n'$ with $\xi\eta\zeta \not\equiv 0 \pmod 3$ is associated uniquely a solution $(\xi, \eta', \zeta')$ of $\gamma = n'$ with $\eta' \not\equiv \zeta' \pmod 3$ and conversely. Therefore $A(n')$ with $\xi\eta\zeta \not\equiv 0 \pmod 3$ is equal to $2C(n')$ with $\eta' \not\equiv \zeta' \pmod 3$.

Suppose $\alpha$ represents $n'$ with $\xi\eta\zeta \equiv 0 \pmod 3$. Then either $\eta^2 + 2\zeta^2 \equiv 0 \pmod 3$ or $\xi^2 + 2\zeta^2 \equiv 0 \pmod 3$ which implies that $\zeta$ and either $\eta$ or $\xi$ respectively are $\equiv 0 \pmod 3$. We reduce the number of solutions by half if we consider only those for which $\xi \not\equiv 0 \pmod 3$. Then, using substitution (10), we have that with every solution $(\xi, \eta, \zeta)$ of $\alpha = n'$ with $\eta \equiv 0 \equiv \zeta \pmod 3$ is associated uniquely a solution $(\xi, \eta', \zeta')$ of $\gamma = n'$ with $\eta' \equiv \zeta' \pmod 3$ and conversely.

This gives

$$C(n') = \tfrac{1}{2}A(n') \quad \text{if } n' \equiv 1 \pmod 3. \tag{11}$$

Then from (7) and (11) we have

$$B(3n') = \tfrac{1}{2}A(9n') \text{ if } n' \equiv 1 \pmod 3. \tag{12}$$

6. *The final formulae.* Using relationships (2) and (3) with (5), (8), (9), (12) we have with (6) the following relationships between $N$ and $B$:

$$\begin{aligned} B(3n') &= \tfrac{1}{2}\tau\{N(18n') - N(2n')\}, \text{ if } n' \equiv 2 \pmod 3, \\ B(9n'') &= B(n'') + \tfrac{1}{2}\tau\{N(54n'') - N(6n'')\}, \\ B(3n') &= \tfrac{1}{2}\tau N(18n'), \text{ if } n' \equiv 1 \pmod 3, \end{aligned}$$

where $\tau = 1/3$ or $1$ according as $n$ is odd or even.

It may be noted that (11), together with the relationships $C(3n) = B(n)$ and $C(n) = 0$ if $n \equiv 2 \pmod 3$ noted in the previous section, give us, in view of the results above, the complete solution of the problem for the form $\gamma$.

The reader may also easily deduce relationships between the proper repre-

sentations by the various forms, i.e. where not all three variables have a factor in common.

After this paper was in type, an article by Uspensky appeared in the American Journal of Mathematics, vol. 51 (1929), p. 51, in which he obtained complete results for the forms here dealt with. This paper, however, goes more into detail concerning the explicit relationships involved, and comparison of corresponding results in the two papers gives certain interesting class number relationships.

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## CONJUGATE LINES ON A SURFACE

By P. J. FEDERICO, Washington, D. C.

1. *Introduction.* A system of conjugate lines on a surface consists of two families of curves such that the tangents to a curve from each family at their points of intersection are conjugate tangents to the surface at that point; that is, the tangents are parallel to conjugate diameters of the Dupin indicatrix of the surface at that point. Conjugate lines possess the property that the tangents to the curves of one family at their points of intersection with a curve of the other family form a developable surface. These developable surfaces are circumscribed about the surface, the line of tangency of one being a curve of one family and the generators being tangent to the curves of the other family. Obviously, for each conjugate system on a surface there are two families of these circumscribed developables.[1]

If the equations of a surface are in the form

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \tag{1}$$

the equations of the tangents to the parametric curves are

$$\frac{(X-x)}{\left(\frac{\partial x}{\partial u}\right)} = \frac{(Y-y)}{\left(\frac{\partial y}{\partial u}\right)} = \frac{(Z-z)}{\left(\frac{\partial z}{\partial u}\right)}, \tag{2}$$

$$\frac{(X-x)}{\left(\frac{\partial x}{\partial v}\right)} = \frac{(Y-y)}{\left(\frac{\partial y}{\partial v}\right)} = \frac{(Z-z)}{\left(\frac{\partial z}{\partial v}\right)}, \tag{3}$$

with $X$, $Y$, $Z$ as running coordinates. These are the equations of two one-parameter families of ruled surfaces. The condition that these ruled surfaces be developable is

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[1] L. P. Eisenhart, *Differential Geometry*, pp. 126–128.

$$(4)\qquad \begin{vmatrix} \dfrac{\partial^2 x}{\partial u \partial v}, & \dfrac{\partial^2 y}{\partial u \partial v}, & \dfrac{\partial^2 z}{\partial u \partial v} \\ \dfrac{\partial x}{\partial u}, & \dfrac{\partial y}{\partial u}, & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v}, & \dfrac{\partial y}{\partial v}, & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0.$$

From the properties of determinants it is seen that $x$, $y$, and $z$ each satisfy an equation

$$(5)\qquad \frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} = 0, \quad \theta = x, y, z,$$

where $a$ and $b$ are functions of $u$ and $v$. Either (4) or (5) is the necessary and sufficient condition that the parametric lines on a surface be conjugate. If $x$, $y$, and $z$, functions of $u$ and $v$, are three linearly independent solutions of a partial differential equation of the type (5), they define a surface upon which the parametric lines form a conjugate system.[1]

Equation (5) is called the point equation of the surface and its invariants are

$$(6)\qquad h = \frac{\partial a}{\partial u} + ab, \quad k = \frac{\partial b}{\partial v} + ab.$$

2. The purpose of this paper is to find the equations of those surfaces for which the circumscribing developables are cylinders or cones.

If the developables (2) are cylinders, the direction cosines of the generators for each cylinder must be constant. In (2), $u$ is constant for each generator and hence $\partial x/\partial u$, $\partial y/\partial u$ and $\partial z/\partial u$ must, save for a common factor, be independent of $v$; that is

$$(7)\qquad \frac{\partial x}{\partial u} = f(u,v)U_1, \quad \frac{\partial y}{\partial u} = f(u,v)U_2, \quad \frac{\partial z}{\partial u} = f(u,v)U_3,$$

where $U_1$, $U_2$ and $U_3$ are functions of $u$ alone. Integration of (7) shows that the equations of the surface are of the form

$$(8)\quad x = \int U_1 f(u,v)du + V_1, \; y = \int U_2 f(u,v)du + V_2, \; z = \int U_3 f(u,v)du + V_3,$$

where $V_1$, $V_2$ and $V_3$ are functions of $v$ alone. The circumscribing developables of (8) whose lines of contact are the curves $v =$ constant are cylinders.

Differentiation of (7) with respect to $v$ gives

$$(9)\qquad \frac{\partial^2 x}{\partial u \partial v} = \frac{\partial f}{\partial v}U_1, \quad \frac{\partial^2 y}{\partial u \partial v} = \frac{\partial f}{\partial v}U_2, \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial f}{\partial v}U_3.$$

The elimination of $U_1$, $U_2$ and $U_3$ between (7) and (9) gives the three equations

[1] Eisenhart, pp. 195–198.

$$\frac{\partial^2\theta}{\partial u\partial v} - \frac{1}{f}\frac{\partial f}{\partial v}\frac{\partial \theta}{\partial u} = 0, \quad \theta = x,y,z. \tag{10}$$

This equation is of the type (5); it is seen that $b=0$, and that $a=-(\partial f/\partial v)/f$.

Conversely, if $b$ in (5) be taken as zero this equation may be written

$$(\partial/\partial v)\log(\partial\theta/\partial u) = -a.$$

This can be integrated directly, giving

$$x = \int U_1 e^{-\int a dv} du + V_1, \quad y = \int U_2 e^{-\int a dv} du + V_2, \quad z = \int U_3 e^{-\int a dv} du + V_3,$$

which are equivalent to (8).

If the developables (3) are cylinders, it can be shown in the same manner that $a=0$. Hence: *A necessary and sufficient condition that, if the parametric curves are conjugate, one family of circumscribing developables be a family of cylinders is that either a or b is zero.*

Since one family of a conjugate system may be chosen arbitrarily it can always be chosen so that its circumscribing developables are cylinders; the surface is still general. It is not until conditions are imposed upon both families that the surface becomes restricted.

3. If both families of developables are cylinders, $a$ and $b$ are both zero; equation (5) becomes

$$(\partial^2\theta/\partial u\partial v) = 0, \tag{11}$$

and the surface is the surface of translation

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3. \tag{12}$$

Hence the theorem: *Surfaces of translation are the only surfaces that have a system of conjugate lines such that the circumscribing developables to the curves of both families of the system are cylinders.*

4. Equation (2) may be written

$$X = x + w\frac{\partial x}{\partial u}, \quad Y = y + w\frac{\partial v}{\partial u}, \quad Z = z + w\frac{\partial z}{\partial u}. \tag{13}$$

If these developables are cones the generators of each cone must pass through a point, the vertex of the cone. The coordinates of this point are constant for each developable and hence must be independent of $v$. Let $w_0$ be the value of $w$ at this point; then,

$$\frac{\partial X}{\partial v} = \frac{\partial x}{\partial v} + w_0\frac{\partial^2 x}{\partial u\partial v} + \frac{\partial w_0}{\partial v}\frac{\partial x}{\partial u} = 0,$$

$$\frac{\partial Y}{\partial v} = \frac{\partial y}{\partial v} + w_0\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial w_0}{\partial v}\frac{\partial y}{\partial u} = 0,$$

$$\frac{\partial Z}{\partial v} = \frac{\partial z}{\partial v} + w_0\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial w_0}{\partial v}\frac{\partial z}{\partial u} = 0.$$

Or

$$\frac{\partial^2\theta}{\partial u\partial v}+\frac{1}{w_0}\frac{\partial w_0}{\partial v}\frac{\partial\theta}{\partial u}+\frac{1}{w_0}\frac{\partial\theta}{\partial v}=0,\quad \theta=x,y,z. \tag{14}$$

On comparing this with (5) it is seen that

$$a=\frac{1}{w_0}\frac{\partial w_0}{\partial v},\quad b=\frac{1}{w_0}. \tag{15}$$

The equations

$$X=x+\frac{1}{b}\frac{\partial x}{\partial u},\quad Y=y+\frac{1}{b}\frac{\partial y}{\partial u},\quad Z=z+\frac{1}{b}\frac{\partial z}{\partial u}, \tag{16}$$

are, therefore, the equations of the locus of the apices of the family of circumscribed cones. This curve is the degenerate focal surface of the congruence of tangents to the curves $v=$constant.[1]

From (15) it is evident that

$$k=(\partial b/\partial v)+ab=0. \tag{17}$$

In view of (17), equation (5) may be written[2]

$$(\partial\theta_1/\partial v)+a\theta_1=0,\qquad \theta_1=(\partial\theta/\partial u)+b\theta,$$

which can be integrated directly, giving the equations of the surface.

It can be shown in a similar manner that, if the other family of developables are cones,

$$h=(\partial a/\partial u)+ab=0. \tag{18}$$

Hence the theorem: *A necessary and sufficient condition that, if the parametric lines of a surface are conjugate, one family of circumscribing developables be a family of cones is that either one of the two invariants of the point equation of the surface be zero.*

5. If one family of circumscribing developables consists of cylinders and the other of cones, $a=0$, $\partial b/\partial v=0$; $b=f(u)$. Equation (5) becomes

$$(\partial^2\theta/\partial u\partial v)+f(u)(\partial\theta/\partial v)=0, \tag{19}$$

and the equations of the surface are of the form

$$x=\phi(u)[U_1+V_1],\quad y=\phi(u)[U_2+V_2],\quad z=\phi(u)[U_3+V_3],\quad \phi(u)=e^{-\int b\,du}. \tag{20}$$

Surfaces of this type are discussed by Darboux.[3]

6. If both families of circumscribing developables are cones, $h$ and $k$ are both zero,

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[1] Eisenhart, pp. 404–406.

[2] Darboux, G., *Théorie Générale des Surfaces*, Vol. II, (2nd ed., 1915), pp. 23, 24.

[3] Darboux, Vol. I, (2nd ed., 1914), pp. 181–184.

$$h = (\partial a/\partial u) + ab = 0, \quad k = (\partial b/\partial v) + ab = 0; \tag{21}$$

and hence

$$(\partial a/\partial u) = (\partial b/\partial v). \tag{22}$$

Equation (22) is the condition that $a$ and $b$ are the partial derivatives of a function $\lambda$ with respect to $v$ and $u$ respectively. Let $a=(\partial\lambda/\partial v)$, $b=(\partial\lambda/\partial u$; then each of equations (21) reduces to

$$(\partial^2\lambda/\partial u\partial v) + (\partial\lambda/\partial u)(\partial\lambda/\partial v) = 0.$$

This may be written

$$(\partial/\partial u)\log(\partial\lambda/\partial v) = -(\partial\lambda/\partial u),$$

whose integral is $\lambda=\log(U+V)$, where $U$ is a function of $u$ alone and $V$ of $v$ alone. Hence

$$a = V'/(U+V), \quad b = U'/(U+V). \tag{23}$$

Equation (5) now becomes

$$\frac{\partial^2\theta}{\partial u\partial v} + \frac{V'}{U+V}\frac{\partial\theta}{\partial u} + \frac{U'}{U+V}\frac{\partial\theta}{\partial v} = 0, \quad \theta = x,y,z. \tag{24}$$

This equation can be integrated readily, giving

$$x = \frac{U_1+V_1}{U+V}, \quad y = \frac{U_2+V_2}{U+V}, \quad z = \frac{U_3+V_3}{U+V}, \tag{25}$$

as the equations of the surface having its parametric lines conjugate and having both families of circumscribing developables cones.

The loci of the vertices of the cones are the two curves

$$\begin{aligned} &x_1 = (U_1'/U'), \quad y_1 = (U_2'/U'), \quad z_1 = (U_3'/U'), \\ &x_2 = (V_1'/V'), \quad y_2 = (V_2'/V'), \quad z_2 = (V_3'/V'), \end{aligned} \tag{26}$$

whose equations are obtained from (16).

7. Three types of surfaces, the surfaces of translation (12), the surfaces (20), and the surfaces (25), having conjugate systems whose circumscribing developables are cylinders and cones have been derived. In the first both families consist of cylinders, in the second one of cylinders and one of cones, and in the third both consist of cones. The third class of these surfaces obviously includes the other two, the second being the special case when either $U$ or $V$ in (25) is constant and the first the special case when both $U$ and $V$ are constant.

A simple example of the general type of surface will now be given. If

$$\begin{aligned} &U_1 = a(1-n^2)^{1/2}\sin u, \quad U_2 = 0, && U_3 = c\cos u, \\ &V_1 = 0, && V_2 = b(1-n^2)^{1/2}\sinh v, \quad V_3 = cn\cosh v, \\ &U = n\cos u, && V = \cosh v, \quad (n \neq 0), \end{aligned}$$

equations (25) become

$$(27)\quad x=a\frac{(1-n^2)^{1/2}\sin u}{n\cos u+\cosh v},\qquad y=b\frac{(1-n^2)^{1/2}\sinh v}{n\cos u+\cosh v},\qquad z=c\frac{\cos u+n\cosh v}{n\cos u+\cosh v}.$$

These are parametric equations of

$$(28)\qquad (x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1.$$

Since $n$ is an arbitrary constant that is eliminated when (27) are reduced to (28), a surface of the second degree is a surface of the type (25) in an infinite number of ways. The curves (26) are the two straight lines

$$x_1=-a(1-n^2)^{1/2}n^{-1}\cot u,\quad y_1=0,\quad z_1=c/n;$$
$$x_2=0,\quad y_2=b(1-n^2)^{1/2}\coth v,\quad z_2=cn;$$

which are at right angles to each other and at a distance $c(n-n^{-1})$ apart.

Incidentally, it is to be noted that the parametric lines of (27) are plane curves, which shows that surfaces of the second degree have an infinite number of plane conjugate systems of curves.

8.[1] The surfaces (25) may be used as the starting point in the discussion of plane conjugate systems of curves on a surface. Consider a surface $S$ with conjugate lines $C$ and their circumscribing developables $D$. If $S$ is subjected to a transformation by reciprocal polars, $S$ goes into a surface $S_1$, the developables $D$ into the conjugate lines $C_1$ on $S_1$ and the lines $C$ go into the developables $D_1$ of $S_1$.[2] For a surface of the type (25) the developables are cones and, since cones go into plane curves, the polar reciprocal of a surface (25) is a surface with a system of plane conjugate lines.

The equations of the polar reciprocal of (25) are

$$\begin{aligned}
Dx &= (U+V)(U_2'V_3'-U_3'V_2')+U'[V_3'(U_2+V_2)-V_2'(U_3+V_3)]\\
&\quad +V'[U_2'(U_3+V_3)-U_3(U_2+V_2)],\\
Dy &= (U+V)(U_3'V_1'-U_1'V_3')+U'[V_1'(U_3+V_3)-V_3'(U_1+V_1)]\\
&\quad +V'[U_3'(U_1+V_1)-U_1(U_3+V_3)],\\
Dz &= (U+V)(U_1'V_2'-U_2'V_1')+U'[V_2'(U_1+V_1)-V_1'(U_2+V_2)]\\
&\quad +V'[U_1'(U_2+V_2)-U_2(U_1+V_1)],
\end{aligned}\qquad (29)$$

where

$$D=\begin{vmatrix} U_1+V_1, & U_1', & V_1'\\ U_2+V_2, & U_2', & V_2'\\ U_3+V_3, & U_3, & V_3'\end{vmatrix}.$$

These are the general equations of a surface with plane parametric conjugate lines.

The partial derivatives of (29) with respect to $u$ and $v$ are:

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[1] Compare Darboux, Vol. I, pp. 176–180.

[2] Eisenhart, pp. 202, 203.

$$\partial x/\partial u = A_2V_3' - A_3V_2', \quad \partial x/\partial v = B_2U_3' - B_3U_2',$$
$$\partial y/\partial u = A_3V_1' - A_1V_3', \quad \partial y/\partial v = B_3U_1' - B_1U_3',$$
$$\partial z/\partial u = A_1V_2' - A_2V_1', \quad \partial z/\partial v = B_1U_2' - B_2U_1',$$

where the $A$'s and $B$'s have been written for the respective coefficients. If the second and third derivatives are computed it is seen that

$$\begin{vmatrix} x_i', & y_i', & z_i' \\ x_i'', & y_i'', & z_i'' \\ x_i''', & y_i''', & z_i''' \end{vmatrix} = 0, \quad i = u, v,$$

which proves that the parametric lines of (29) are plane.

The curves (26) associated with the surface (25) go into the families of planes

$$U_1'x + U_2'y + U_3'z + U' = 0, \tag{30}$$

$$V_1'x + V_2'y + V_3'z + V' = 0, \tag{31}$$

by a polar reciprocal transformation. These are the planes that contain the parametric lines of (29), the planes (30) contain the curves $v$ = constant and the planes (31) the curves $u$ = constant. The same result may be obtained by multiplying $x$, $y$, and $z$ in (29) by $U_1'$, $U_2'$, and $U_3'$ respectively and adding, and by $V_1'$, $V_2'$, and $V_3'$ and adding.

Let the equation of the tangent plane of (29) be $ax+by+cz+d=0$. Each point $(u, v)$ of (29) lies on a plane of (30) and on a plane of (31) and also on the tangent plane. Hence the surface is the locus of the intersections of these three planes. This intersection is given by

$$x = \frac{d(U_2'V_3' - V_2'U_3') + U'(cV_2' - bV_3') + V'(bV_3' - cU_2')}{|\,a,\ U_2',\ V_3'\,|}$$

and similar expressions for $y$ and $z$. On comparing these equations with (29), it is seen that, within a common factor,

$$a = U_1 + V_1, \ b = U_2 + V_2, \ c = U_3 + V_3, \ d = U + V;$$

and consequently the equation of the tangent plane is

$$(U_1 + V_1)x + (U_2 + V_2)y + (U_3 + V_3)z + (U + V) = 0.$$

## ALGORITHMS FOR THE SOLUTION OF THE QUADRATIC CONGRUENCE[1]

By H. S. VANDIVER, University of Texas

The subject of this paper is an elementary problem of long standing in the history of the theory of numbers. If we consider the congruence

$$x^2 \equiv q \pmod{p}, \tag{1}$$

[1] Presented to the American Mathematical Society at Ithaca, N. Y., Sept. 1925.

where $q$ and $p$ are rational integers, $p$ a prime, and $q$ prime to $p$, then by the use of the law of quadratic reciprocity we have a definite algorithm not involving trial for ascertaining if integral values of $x$ exist satisfying the congruence. If we consider, however, the problem of determining the value of $x$ in this relation no such convenient procedure has as yet been proposed.

In the elementary theory of binary quadratic forms a method is described for finding the values of $x$ and $y$ in the relation

$$ax^2 + bxy + cy^2 = m,$$

all these symbols representing rational integers. *In this connection it should be noted that in order to carry out this method it is always necessary to know the solution of a congruence of the form*

$$x^2 \equiv k \pmod{m}.$$

We shall here discuss a method for the determination of $x$ in (1) which, however, is not devoid of a certain amount of trial. Its relation to former methods which have been published will also be considered.

As there is a certain analogy between known methods of solving the linear congruence

$$ax \equiv 1 \pmod{p}, \tag{2}$$

where $a$ is an integer prime to $p$ and the solutions of (1), we shall describe briefly the former methods.

1. A root of (2) is given by $a^{p-2}$, using Fermat's theorem.

2. A solution of (1) is obtained by expanding $p/a$ as a continued fraction and using the relation $pk+al=\pm 1$, where $l/k$ is the penultimate convergant to 0 to $p/a$.

3. In a previous paper[1] I proposed a method for the solution of (2) which, in the case of prime modulus, may be put in the following form.

Set $k_1p=aq_1+r_1$, where $|r_1|$ is less than $a$ and $k_1$ is a convenient integer chosen to make $|r_1|$ relatively small.

Proceed in a similar way with $p$ and $r_1$ so that $k_2p=r_1q_2+r_2$, where $|r_2|$ is less than $|r_1|$.

We obtain ultimately $k_sp=r_{s-1}q_s+r_s$, where $r_s=\pm 1$. Considering these expressions modulo $p$ we may write $r_{i-1}q_i \equiv -r_i \pmod{p}$. Setting $i=1, 2, \cdots, s$, and multiplying all the congruences together we find

$$x = (-1)^s \prod_i q_i/r_s.$$

A solution of (1) which has to a certain extent analogy with the first method described above is the solution of (2) which was given by Cipolla.[2] In this a formula for $x$ is given as follows:

[1] This Monthly, vol. 31 (1924), pp. 137–140.

[2] Mathematische Annalen, vol. 63 (1907), pp. 54–61. A somewhat different formula for the case $p=8n+1$ ($n$ odd) was given by Caris, in this Monthly, vol. 32 (1925), pp. 294–7.

$$x = \frac{-1}{2^{r-2}} q^{\lambda-1} \sum_{s=0} q^{\mu}/(\omega^{\nu} - 1), \tag{3}$$

where

$$\lambda = (p + 2^r - 1)/2^{r+1}; \quad \mu = s(p-1)/2^r; \quad \nu = (2s+1)(p-1)/2^r;$$

and where $\omega$ is a quadratic non-residue of $p$, and $2^r$ is the highest power of 2 dividing $p-1$.

For $p \equiv 3 \pmod 4$, this reduces to the well known solution $x \equiv q^{(p+1)/4}$, and if $p \equiv 5 \pmod 8$, we may take $\omega = 2$ and obtain

$$x \equiv \tfrac{1}{2} q^{(p+3)/8} \{2^t + 1 - (2^t - 1) q^t\}, \quad t = (p-1)/4.$$

Note that this solution involves the element of trial in the selection of $\omega$ and that (3) becomes very complicated when $r$ is large. Kummer[1] gave a proof of the law of quadratic reciprocity (involving, however, an unproved assumption; see the top of p. 14 of his article). His argument may be modified in such a way as to give an algorithm for the determination of $x$ in (1), if $q$ is prime.

If we examine the Pellian equation

$$t^2 - Du^2 = 1, \tag{4}$$

where $D$ is a positive integer of the form $4n+1$, then $t$ is odd and $u$ is even, and we have integers $m$ and $m'$ so that

$$t + 1 = 2m\delta^2, \quad t - 1 = 2m'\lambda^2,$$

in which $mm' = D$ and $2\delta y = u$. Elimination of $t$ gives

$$m\delta^2 - m'\lambda^2 = 1. \tag{5}$$

Assume now that $D = pq$, and $p$ and $q$ are primes of the form $4n+3$ and that $(t, u)$ is a solution of (4) in which $x$ and $y$ are the smallest possible positive integers. In view of the latter assumption we can not have $1 = \delta^2 - pq\lambda^2$. Also since $p$ is a prime of the form $4n+3$ the relation $1 = pq\delta^2 - \lambda^2$ does not exist. Hence if $(q/p) = 1$, then (5) gives $1 = q\delta^2 - p\lambda^2$, or $q\delta^2 \equiv 1 \pmod p$, and $x \equiv \pm 1/\delta \pmod p$, for the solution of (1).

*Example*: If $x^2 \equiv 3 \pmod{71}$, we obtain by expanding square root of 213 as a continued fraction, or from existing tables, $194399^2 - 213 \cdot 13320^2 = 1$, whence (5) becomes $1 = 3 \cdot 180^2 - 71 \cdot 37^2$, and $28^2 \equiv 3 \pmod{71}$.

The above method is devoid of the element of trial, but this is not the case if we use an analogous method for $p$ a prime of the form $4n+1$ and $q$ is a prime of the form $4n+3$. Here we have to select an auxiliary prime $p'$ and put $D = pp'q$, $p'$ being selected by trial to satisfy the conditions $(p'/p) = -1$, $(p'/q) = -1$. However, after $p'$ has been found, the method is devoid of trial although the work will be very long in many cases since it is known that the values of $t$ and $u$ in (4) are quite large even for small values of $D$. Obviously

[1] Journal für Mathematik, vol. 100 (1887), pp. 10–14.

we are determining integers $\delta$ and $\lambda$ which satisfy the relation $1=q\delta^2-p\lambda^2$, a relation which is more special than (1).

In particular cases the computation may become relatively short as it sometimes happens that there are only four or five terms in a period in the development of $\sqrt{D}$ as a continued fraction. For example, consider $x^2\equiv -4019 \pmod{9043}$.

The method which is described for the solution of (1) obviously has a type or analogy with the second solution of (2). *This leads us to inquire as to the possibility of there being a method for the solution of* (1) *which is analogous to the third method described for the solution of* (2). So far this investigation has yielded nothing.

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## QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

*The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

### DISCUSSIONS

### I. A NOTE ON THE HYDRODYNAMICAL THEOREM ON THE RELATION BETWEEN IMPULSE AND ENERGY

By NRIPENDRANATH SEN, Calcutta University

The well-known hydrodynamical theorem on the relation between impulse and energy viz.,

$$\xi,\eta,\zeta,\lambda,\mu,\nu = \frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}, \frac{\partial T}{\partial w}, \frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}, \text{ respectively}$$

where $u, v, w, p, q, r$ are six components of the motion of the solid, $\xi, \eta, \zeta, \lambda, \mu, \nu$ are six components of the impulse, and $T$ is the kinetic energy of the solid and liquid, has been established in many standard books on the subject, including that of Professor Lamb,[1] on the supposition that the components of motion satisfy the relations

$$\frac{\delta u}{u} = \frac{\delta v}{v} = \frac{\delta w}{w} = \frac{\delta p}{p} = \frac{\delta q}{q} = \frac{\delta r}{r}.$$

The object of the present note is to supply the following rigorous proof of the above theorem.

Since $\xi, \eta, \zeta, \lambda, \mu, \nu$ are linear in $u, v, w, p, q, r$ the latter may be expressed as linear functions of the former (See Lamb, §123, equation 6). Hence, $T$ may be

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[1] Horace Lamb, *Hydrodynamics*, (5th edition), p. 157.

regarded as a homogeneous quadratic function of $\xi, \eta, \zeta, \lambda, \mu, \nu$. Denoting it by $T'$ when thus expressed, we have by the principle of energy,

$$(1) \qquad u\delta\xi + v\delta\eta + w\delta\zeta + p\delta\lambda + q\delta\mu + r\delta\nu = \delta T$$
$$= \delta T' = \frac{\partial T'}{\partial \xi}\delta\xi + \frac{\partial T'}{\partial \eta}\delta\eta + \frac{\partial T'}{\partial \zeta}\delta\zeta + \frac{\partial T'}{\partial \lambda}\delta\lambda + \frac{\partial T'}{\partial \mu}\delta\mu + \frac{\partial T'}{\partial \nu}\delta\nu.$$

Therefore

$$(2) \qquad u, v, w, p, q, r = \frac{\partial T'}{\partial \xi}, \frac{\partial T'}{\partial \eta}, \frac{\partial T'}{\partial \zeta}, \frac{\partial T'}{\partial \lambda}, \frac{\partial T'}{\partial \mu}, \frac{\partial T'}{\partial \nu}.$$

Also, since $T'$ is a homogeneous quadratic function of $\xi, \eta, \zeta, \lambda, \mu, \nu$,

$$2T = 2T' = \xi\frac{\partial T'}{\partial \xi} + \eta\frac{\partial T'}{\partial \eta} + \zeta\frac{\partial T'}{\partial \zeta} + \lambda\frac{\partial T'}{\partial \lambda} + \mu\frac{\partial T'}{\partial \mu} + \nu\frac{\partial T'}{\partial \nu}$$
$$= \xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r, \text{ by (2)}.$$

Performing an arbitary variation $\delta$ and omitting terms which cancel by (1), we find

$$\xi\delta u + \eta\delta v + \zeta\delta w + \lambda\delta p + \mu\delta q + v\delta r$$
$$= \delta T = \frac{\partial T}{\partial u}\delta u + \frac{\partial T}{\partial v}\delta v + \frac{\partial T}{\partial w}\delta w + \frac{\partial T}{\partial p}\delta p + \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial r}\delta r,$$

whence

$$(3) \qquad \xi, \eta, \zeta, \lambda, \mu, \nu = \frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}, \frac{\partial T}{\partial w}, \frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}.$$

## II. Rationalizing Factors and the Method of Undetermined Coefficients

By L. J. Paradiso, Cornell University

In elementary algebra the question is sometimes raised by a student as to how a rationalizing factor can be obtained for an expression which contains a root higher than a square root. For example, to rationalize the denominator of $2/((2)^{1/3}+(3)^{1/2})$. In many cases a rationalizing factor is easily obtained by the use of the conjugate, as in the example $(a^{1/2}+b^{1/2})(a^{1/2}-b^{1/2})=a-b$ or by using a factor of an expression such as $x^n+y^n$. For example, a rationalizing factor of $2(3)^{1/3}-5(5)^{1/3}$ is $(2(3)^{1/3})^2+10(15)^{1/3}+(5(5)^{1/3})^2$; it is obtained by using $x=2(3)^{1/3}$, $y=5(5)^{1/3}$ and $x^3-y^3=(x-y)(x^2+xy+y^2)$. But in rationalizing expressions of the form

$$a_0 + a_1(p)^{1/n} + a_2(p)^{2/n} + \cdots + a_{n-1}(p)^{(n-1)/n},$$

where the $a_i$ are rational coefficients, the process usually given is not readily remembered nor easy to teach without a background of the general theory.[1]

[1] See G. Chrystal, Algebra, Part 1, (1878), p. 197.

It is the object of this note to show that theoretically in all cases and practically in many cases a rationalizing factor may be found by the method of undetermined coefficients. Although this method is taught to freshmen in college in the case of partial fractions, it is not commonly employed to find rationalizing factors.

The theorem which we shall use to justify the method of undetermined coefficients is: *If $\phi$ is an algebraic number of degree $m$, that is, the root of a uniquely determined equation $f(x)=0$ of degree $m$ with rational coefficients, and if $R(\phi)$ is a rational function of the algebraic number that is, $R(\phi)=g(\phi)/h(\phi)$, then*

$$R(\phi) = r_0 + r_1\phi + r_2\phi^2 + \cdots + r_{m-1}\phi^{m-1},$$

*where the $r_i$ are rational coefficients.*

Suppose we wish to find a rationalizing factor for

$$a_0 + a_1\phi + \cdots + a_{m-1}\phi^{m-1},$$

where the $a_i$ are rational numbers. The problem amounts to finding rational numbers $A_i$ such that

$$(A_0 + A_1\phi + A_2\phi^2 + \cdots + A_{m-1}\phi^{m-1})(a_0 + a_1\phi + \cdots + a_{m-1}\phi^{m-1}) = 1.$$

Multiplying the factors in the left hand member of the above and arranging the terms, we can equate the constant term and the coefficients of $\phi$, $\phi^2$, $\cdots$, $\phi^{m-1}$ respectively to $1, 0, 0, \cdots$. In this way we obtain $m$ simultaneous equations in the $A_i$ from which we can find the required coefficients.

Example 1: To find the rationalizing factor of $2(2)^{1/3}-(4)^{1/3}-2$, we let $\phi=(2)^{1/3}$ so that $\phi^3=2$. We then have $(A_0+A_1\phi+A_2\phi^2)(2\phi-\phi^2-2)=1$. We group the terms

$$(4A_2 - 2A_1 - 2A_0) + (-2A_2 - 2A_1 + 2A_0)\phi + (2A_1 - 2A_2 - A_0)\phi^2 = 1,$$

from which we get

$$4A_2 - 2A_1 - 2A_0 = 1,\ 2A_0 - 2A_1 - 2A_0 = 0,\ 2A_1 - 2A_2 - A_0 = 0,$$

from which we find the rationalizing factor to be

$$-(2/5) - (3/10)\cdot 2^{1/3} - (1/10)\cdot 4^{1/3}.$$

By this same method we can of course rationalize numerators as well as denominators.

We can generalize this method to cases where more than one algebraic number is involved in the expression to be rationalized; say $\phi$ of degree $m$ and $\theta$ of degree $n$. In this case in forming the rationalizing factor with undetermined coefficients one must not only put undetermined coefficients with powers of $\phi$ up to $m-1$ and powers of $\theta$ up to $n-1$, but also one must attach undetermined coefficients to all possible different cross products which can be formed from the various powers of $\phi$ and $\theta$ using the restriction,[1] of course, that $\phi^m=\phi$ and

[1] Loc. cit., p. 192, II.

$\theta^n = \theta$. This can be further generalized to any number of algebraic numbers, but in most cases where the index of the root is greater than 3 and the number of different algebraic numbers is more than three it becomes impractical to solve the simultaneous equations obtained.

Example 2, illustrating a more general case: Rationalize the denominator of $1/((a)^{1/2}+(b)^{1/3})$ where $a$ and $b$ are rational. Let $x=(a)^{1/2}$, $x^2=a$, and $y=(b)^{1/3}$, $y^3=b$. Then

$$1/(x+y) = A + Bx + Cy + Dy^2 + Exy + Fxy^2.$$

Expanding, arranging the terms, and equating the coefficients we have

$$aB + bD = 1,\ B + C = 0,\ D + E = 0,\ C + aF = 0,\ A + aE = 0,\ A + bF = 0$$

from which we get

$$1/(a^{1/2} + b^{1/3}) = \frac{ab + a^2 \cdot a^{1/2} - a^2 \cdot b^{1/3} + b \cdot b^{2/3} - b \cdot a^{1/2} \cdot b^{1/3} - a \cdot a^{1/2} \cdot b^{2/3}}{a^3 + b^2}.$$

## III. A Note on the Differential Equations of Geodesics

By B. F. Kimball, Cornell University

The differential equations of geodesics on a surface in parametric form with the arc-length $s$ taken as parameter are often given in three ways. Let $u$ and $v$ be the curvilinear coordinates of a point on the surface,

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

and let $H=(EG-F^2)^{1/2}$. We consider only a region on the surface for which $H>0$. Let primes denote differentiation with respect to $s$ and subscripts differentiation with respect to variable indicated by the subscript. Taking

$$(1)\qquad \begin{aligned} P &= Eu'' + Fv'' + \tfrac{1}{2}E_u u'^2 + E_v u'v' + (F_v - \tfrac{1}{2}G_u)v'^2, \\ Q &= Fu'' + Gv'' + (F_u - \tfrac{1}{2}E_v)u'^2 + G_u u'v' + \tfrac{1}{2}G_v v'^2, \end{aligned}$$

the three forms in which the differential equations of geodesics are often given are:

$$(2)\qquad P = 0,\quad Q = 0,$$

$$(3a)\qquad \frac{d(Eu' + Fv')}{ds} - (\tfrac{1}{2}E_u u'^2 + F_u u'v' + \tfrac{1}{2}G_u v'^2) = 0$$

$$(3b)\qquad \frac{d(Fu' + Gv')}{ds} - (\tfrac{1}{2}E_v u'^2 + F_v u'v' + \tfrac{1}{2}G_v v'^2) = 0,$$

and

$$u'' + \begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix} u'^2 + 2\begin{Bmatrix} 1 & 2 \\ & 1 \end{Bmatrix} u'v' + \begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix} v'^2 = 0, \tag{4a}$$

$$v'' + \begin{Bmatrix} 1 & 1 \\ & 2 \end{Bmatrix} u'^2 + 2\begin{Bmatrix} 1 & 2 \\ & 2 \end{Bmatrix} u'v' + \begin{Bmatrix} 2 & 2 \\ & 2 \end{Bmatrix} v'^2 = 0. \tag{4b}$$

[1]

We note before going further that the expression on the left hand side of equation (3a) is another form of the function $P$ and that the expression on the left hand side of (3b) is $Q$. Also it is true that the expressions on the left hand side of equations (4a) and (4b) are respectively:

$$(PG - QF)/H^2 \quad \text{and} \quad (QE - PF)/H^2. \tag{5}$$

Most writers on differential geometry speak of the fact that in finding the geodesics on a surface it is, under certain conditions, sufficient to solve only one equation from any one pair of equations (2), (3) or (4). For instance all solutions of $P=0$ for which $v$ is not constant are geodesics, and such solutions give all the geodesics on the surface for which $v$ is not constant. None of the writers, however, seems to bring together the following simple relations between the geodesic curvature and the pairs of equations (2), (3), and (4) which enable one to see at a glance what the nature of the interdependence of any two equations belonging to the same pair is. Let $k_g$ denote the geodesic curvature. Under the usual definition one finds that[2]

$$k_g = H^{-1}[(Eu' + Fv')Q - (Fu' + Gv')P]. \tag{6}$$

Differentiating the identity

$$Eu'^2 + 2Fu'v' + Gv'^2 \equiv 1 \tag{7}$$

with respect to $s$ it is not hard to show that

$$Pu' + Qv' \equiv 0. \tag{8}$$

Then from (8) and (6) it is seen that

$$P = -v'Hk_g, \qquad Q = u'Hk_g. \tag{9}$$

Substitute these values of $P$ and $Q$ in (5) and one finds that

$$\begin{aligned} (PG - QF)/H^2 &= -k_gH^{-1}(Fu' + Gv'), \\ (QE - PF)/H^2 &= k_gH^{-1}(Eu' + Fv'). \end{aligned} \tag{10}$$

Relations (9) indicate the nature of the interdependence of the two equations of either of the pairs (2) or (3), and the relations (10) show to what extent solutions of one of the equations (4) can be trusted to give geodesics on the surface. Since $Fu'+Gv'=0$ is the condition that a curve be orthogonal to the

[1] See L. P. Eisenhart, *Differential Geometry*, p. 153, for values of these symbols in terms of $E$, $F$, and $G$ and their derivatives.

[2] See L. P. Eisenhart, *Differential Geometry*, p. 134.

$u=$const. curves and $Eu'+Fv'=0$ is the condition that it be orthogonal to the $v=$const. curves, we conclude that *any solution of* (4a) *which is not orthogonal to* $u=$*const. curves is a geodesic, and that any solution of* (4b) *which is not orthogonal to* $v=$*const. curves is a geodesic.* It is not improbable that the above results are well-known to differential geometers but one does not find the relations (9) and (10) introduced with the differential equations (2), (3), and (4) in the books now available on differential geometry. Bianchi[1] notes the relation (8), and Bolza[2] notes relations similar to (9) taking the calculus of variations point of view. Lilienthal,[3] in discussing geodesic curvature, derives formulae (9) and perhaps will apply them to the discussion of geodesics in the second part of Vol. II which does not seem to be out yet.

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## RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y.

*All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Lennes, N. J.* College Algebra. New York, Harper and Brothers, 1928. xiv+301 pages. $2.25.

*Harding, A. M.* and *Mullins, G. W.* Plane Trigonometry. New York, The Macmillan Company, 1928. viii+118 pages.

*Harding, A. M.* and *Mullins, G. W.* Analytic Geometry. viii+312 pages +14 pages of answers.

*Ford, W. B.* College Algebra, Revised Edition. New York, Henry Holt and Company, 1928. vi+278 pages.

*Ford, W. B.* First Course in the Differential and Integral Calculus. New York, Henry Holt and Company, 1928. viii+372 pages. $3.00.

*Townsend, E. J.*, Functions of Real Variables. New York, Henry Holt and Company, 1928. x+406 pages.

*Brasch, F. E.*, Editor. Sir Isaac Newton, 1727–1927. With an introduction by David Eugene Smith. Baltimore, The Williams and Wilkins Company, 1928. x+352 pages. $5.00.

"A series of essays giving a modern interpretation of the great English physicist through American eyes. Discusses Newton as physicist, mathematician, chemist, astronomer, philosopher, religious thinker, etc."

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[1] Bianchi, *Differentialgeometrie* (1899), p. 153.

[2] Bolza, *Concerning the Geodesic Curvature and the Isometric Problem on a Geometric Surface*, The Decennial Publications of the Univ. of Chicago, First series, vol. IX (1904), p. 13.

[3] Lilienthal, *Vorlesungen über Differentialgeometrie*, vol. II (1913), p. 224.

*Turnbull, H. W.* The Theory of Determinants, Matrices, and Invariants. London and Glasgow, Blackie and Son, Ltd., 1928. xvi+338 pages. 25 s.

"A general outline of invariant theory and related subjects. Both direct and symbolic methods are used. A special feature is the inclusion of recent researches on higher forms than the binary. Seven chapters are devoted to determinants and matrices."

*Knopp, Konrad.* Theory and Application of Infinite Series. Translated from the second German edition by Miss R. C. Young. London and Glasgow, Blackie and Son Ltd., 1928. xii+572 pages. 30 s.

The second German edition was reviewed in this Monthly vol. 32 (1925), p. 382, by Tomlinson Fort. In the English edition there has been added a chapter (62 pages) on Euler's summation formula and asymptotic expansions.

*Schrödinger, Erwin.* Four Lectures on Wave Mechanics. London and Glasgow, Blackie and Son, Ltd., 1928. viii+54 pages. 5 s.

*Fisher, R. A.* Statistical Methods for Research Workers. Second edition. London, Oliver and Boyd, 1928. xiv+270 pages, with 6 tables. 15 s.

One of a series of Biological Monographs and Manuals, and designed primarily for the use of workers in biological science.

*Strachan, Robert C.* A Table of Hyperbolic Radians, adapted to the derivation of hyperbolic functions from trigonometric tables. With explanatory text and notes on the catenary. Published by the author, 75 West St., New York N. Y. 1928. $2.00.

## REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

*Plane Trigonometry.* By Raymond W. Brink. The Century Company, 1928. xii+198 pages.

*Trigonometry, Plane and Spherical.* By David Raymond Curtiss and Elton James Moulton. D. C. Heath & Co., 1927. xi+264 pages.

*Plane Trigonometry.* By N. J. Lennes and A. S. Merrill, with the editorial cooperation of H. E. Slaught. Harper & Brothers, 1928. x+179 pages.

These recent texts in trigonometry help us to believe that better things are in store for the students as well as for the teachers of this subject. We may ask, in considering any trigonometry, the following questions: Should the subject "be regarded primarily as preparation for calculus rather than surveying"? Is the purpose of the course skill in computation, or a knowledge of trigonometric analysis, or both? What minimum amount will meet a student's actual needs? How much will serve his further mathematical training and development? Is the course in trigonometry to be given to students of high school maturity or to freshmen in college who may have had or are having some advanced work in algebra? An examination of these three texts leads one to feel that there is in

each a forward look to the calculus, and an aim to serve the student not merely in computation but in all further training in science and mathematics.

Professor Brink sets forth his own plan and purpose very definitely at the beginning of the preface of his text, "to keep clearly before the student the values which he may expect to receive in return for his efforts." The book begins consistently with a discussion of practical situations and gives a good variety of applied problems throughout. When the student has seen something of the usefulness of the trigonometric functions, has grown somewhat accustomed to the idea of the general angle, with its unrestricted size and double quality value, a more detailed analytical study is made of the functional relations. The final lists of exercises on completing the chapters furnish excellent problem material.

The general angle is used in defining the trigonometric functions; the notation arc sin 1/2 is used in place of the notation $\sin^{-1}1/2$ which has tormented students so long with its seemingly meaningless exponential form; and the full discussion of logarithms, as well as the careful explanation of the solution of trigonometric equations, suggests the advantage of such a text for students with a minimum of algebra.

As has been said, the text aims, in true American fashion, to gain the interest of the students through its practical values, believing this interest will carry over into a more analytical and critical study of the background of the subject.

In the text of Curtiss and Moulton, the first chapter of over 30 pages gives one of the most complete presentations of the general angle to be found in our American text-books in trigonometry. The angle is referred to both rectangular and polar coordinates; the six trigonometric functions are defined in terms of ordinate, abscissa, and distance; and the chapter is concluded with a study of some special angles including not only 30°, 45°, 60°, but also those angles whose functions are numerically the same. There is no hint of suggested values in return for efforts expended but perhaps these would come with a broader viewpoint and stimulated mental energy. A student who had mastered the substance of this chapter would begin to "see what it is all about" and would already be far along in trigonometry.

The solution of triangles both right and oblique follows, and the more difficult work, that of reduction of angles and the relations between changes in and processes on the angle and changes in and processes with the functions. Fundamental identities, radian measure, inverse functions and trigonometric equations complete the course. The question arises whether radian measure and inverse functions would not stand a better chance in later mathematical work if started earlier in trigonometry and used consistently in it. A chapter on logarithms is included although the author suggests the possible omission of computation by logarithms as "in accord with the growing tendency to calculate with slide rules, machines, and multiplication tables."

The trigonometry by Lennes and Merrill is an attractive book to read and to

teach. Starting with the simple definition of trigonometric functions of an acute angle in terms of the sides opposite and adjacent to the angle and the hypotenuse, in a right triangle which may be formed to include the angle, the work is consistently developed to include the general angle with its theory. Certain features of the book should be mentioned: the paging that leaves diagram and proof together; several groups of problems covering the same work, which may be given to different sections of the same class; cumulative reviews at the end of the course; a full chapter on logarithms; and a chapter on the history of trigonometry, suggesting that student and teacher may be helped by seeing the part the subject of trigonometry has played in mathematics and science in the past. Radian measure, variation of functions, inverse functions, and solution of trigonometric equations are relegated to the last chapters. Why not introduce them all earlier and use them as occasion suggests "ever after"? And why a "five-part" theorem in reduction of angles instead of a "one-part"? Our students of trigonometry, and even more our students of calculus, are crying out for a good, single, general theorem in reduction of angles.

HARRIET E. GLAZIER

*Theory of Probability.* By the late William Burnside with a preface and memoir on Burnside by A. R. Forsyth. Cambridge University Press, 1928. xxx +106 pages.

This posthumous work by the distinguished English mathematician William Burnside "is the expression of his views so far as they had been framed into a system," to quote from Forsyth's memoir which precedes Chapter 1. In the preface Forsyth states that Burnside's first paper on the subject was one of 1918, and that the manuscript now published "was written at intervals before the middle of 1925"; in other words, when Burnside was at the ages of sixty-six and seventy-three years, respectively. Long before this his work had placed him among outstanding mathematicians. Therefore we may anticipate that the work reflects the advantages of approaching a difficult subject with mature judgment and a mastery of analysis. On the other hand, the subject could not have received from him the same sustained thought which he gave to the theory of groups and allied subjects.

The outstanding feature of Chapter 1, the Introduction, is the rule given by Burnside for the calculation of probabilities. It ends with the phrase "provided that *each two* of the $n$ results are *assumed* to be equally likely." Burnside contrasts this rule with one given by Poincaré, embodying the phrase "à condition que tous les cas soient également vraisemblables." The distinction is discussed in a note given on page 101 in which two issues are raised:

1. The relative significance of the two phrases "provided that all cases are equally likely" and "provided that all cases are *assumed* to be equally likely."

2. Assuming that *each two* cases are equally likely as against assuming that *all* cases are equally likely.

Regarding the first issue raised, it may be questioned whether Burnside has

interpreted the phrase "à condition que tous les cas soient également vraisemblables" in the sense that Poincaré had in mind when he said "comment reconnaître que tous les cas son également probables? . . . . nous devrons, dans chaque application, faire des *conventions*, dire que nous considérerons tel et tel cas comme également probables," on page 28 of the *Calcul des Probabilités*, second edition.

Burnside gives two groups of examples illustrating the application of his rule and the formulae of the Introduction: those solved by direct methods being placed in Chapter II, while such classic problems as the "duration of play," involving finite difference equations and generating functions, appear in Chapter III. Bridge players should be warned that Burnside's solution of example II, page 16, overlooks the question of mutually exclusive cases. A hand containing a single heart might also contain but a single diamond; therefore, the number of favorable cases is less than four times $(13)\,(39!) \div (27!)\,(12!)$. Example X, page 22, deals with the distribution of points placed at random on a line; it is typical of an important class of problems which are given further consideration in Chapter VI, entitled "Probabilities connected with geometrical questions."

The major subject of Chapter IV is "Methods of approximation"; but also, page 45, "It is convenient here to introduce two conceptions which prove to be of great value in many appplications of the theory of probability; viz. those of the 'probable value,' and the 'most probable value.'"

In Chapter V, "Probability of causes," the Bayes' formula is developed and illustrated by several examples. Burnside brings out clearly the extent to which the solution of an *a posteriori* problem depends on the information existing with reference to the *a priori* probabilities in favor of the possible causes. When such information is lacking the solution of a problem of this type is possible only on the basis of some assumptions. Different sets of assumptions usually lead to different final results so that the weight assignable to each set of assumptions is a question of major importance when the theory is applied to practical problems.

The book closes with two chapters dealing with errors of observations. Inconsistencies within these chapters and also in the cognate papers published by Burnside in the Cambridge Philosophical Society Proceedings call for careful consideration, particularly on account of their bearing on the question of "small samples."

In Article 27, where a set of observations made on an unknown quantity is dealt with, the *a priori* probability function for the unknown true value of the measured quantity is introduced. Therefore, in Article 36, where not only the observed quantity but the precision constant $h$ in the Gaussian law is unknown, one would expect to find an *a priori* function involving both of these unknowns; but no *a priori* function whatever appears and the analysis proceeds with the totally different problem of observations not yet made on a known quantity with a known precision constant. Burnside arrives at a result that is identical

with the formula published by Student in 1908, a formula which is not the answer to the *a posteriori* problem *unless arbitrary assumptions are made* such as:

1. *A priori* all values of the unknown observed quantity are equally likely.

2. *A priori* the probability that the precision constant has a value in the neighborhood of $h$ is inversely proportional to $h$.

The *a posteriori* problem here considered constituted the subject matter of two papers by Burnside entitled, "On errors of observation" and a "Note on Dr. Burnside's recent paper on errors of observation" by R. A. Fisher (listed under "Cognate Papers" on page 104 of the book). In his second paper Burnside retracts the conclusions set down in his first paper; in his note R. A. Fisher is, in the opinion of the reviewer, far from giving a legitimate treatment of the problem. However, we are indebted to Burnside for a short rigorous derivation of the Student formula.

EDWARD C. MOLINA

*Vorlesungen über Differential- und Integralrechnung.* Bd. I. Funktionen einer Veränderlichen. By R. Courant. Berlin, Julius Springer, 1927. xiv+410 pages.

The standpoint and aim of this excellent book are clearly stated in the following freely translated extract from the author's preface.

"Mathematical literature is certainly not poor in good works on the calculus. Nevertheless it is not easy for the beginner to find a book that affords a straight path into the living body of the science and gives an intelligent freedom of motion among its applications. The beginner should not be wearied by diffuseness or lack of content, nor can he support a pedantry which does not recognize the difference between essentials and non-essentials and which—as often in systematized axiomatic treatments—throws a misty veil over the actual driving forces of our science and over its objective kernel.

"It is indeed easier to see and feel defects than to remedy them. I am far from the idea that I can supply the ideal text for the beginner. But I do not believe that the publication of my lectures is superfluous. They differ from current texts in the order and choice of material, in their tendency, and perhaps also in their expository style.

"Most striking, perhaps, is the break with the surviving tradition of separating the differential and integral calculus. This separation, unwarranted by either material or didactic considerations and merely a result of historic accidents, hinders a clear enforcement of the essential point—the connection between definite integral, indefinite integral, and derivative.

"I have tried to give the reader a clear view of the close connection of analysis to its applications and, with all mathematical rigor and precision, to yield to intuition its full prerogative as the original source of mathematical truths. Indeed the exposition of science as a closed system of static truths, without reference to their origin and goal, has an esthetic charm and fulfils a deep philosophical need. But as an exclusive basis for development or as a

didactic principle, the standpoint of science as an abstract logical system turned inward upon itself, is a great danger. The pursuit of mathematical analysis while turning one's back on the applications and on intuition, will result in abandoning our science to the fate of atrophy and dry rot. To protect the learner from a shadowy and all too convenient purism seems to me a problem of the first importance. This, indeed, is not the last purpose of my book.

"It is designed for everybody with an ordinary school preparation who is earnestly concerned with science and its applications, be he student at a university or college of engineering, teacher or engineer. It does not promise to spare the reader the exercise of his own faculties, but leads directly, without hesitation and unnecessary sidestepping, to fruitful and interesting topics, and seeks to aid the understanding not only by giving the proofs step by step, but also by showing the motivation and connection of the entire argument."

The above program has been faithfully followed and systematically carried out. The author's style, vigorous and clear, reminiscent at times of his predecessor, Felix Klein, makes the perusal of this book a delight. On a camping trip in the Rockies last summer it was easily the favorite of a select, if small, library, not yielding even to Hardy's *Return of the Native.* Although the treatment is often informal and always unpedantic, a modern standard of rigor is maintained throughout. The old conflict between vigor and rigor is here at last resolved. If it is not a *Calculus Made Easy* it is at least a *Calculus Made Right.* The author is undoubtedly familiar with both words and music of the old song:

If you would take an epsilon,
  And I should take a delter,
Our dear old mathematics
  Wouldn't be quite so helter skelter.
But if you make it easy,
  By making it quite wrong,
You'll have to learn it all again—
  'Twill take you twice as long.

In the introduction the author contrasts the old mathematics with the new. While the former is based on geometry, even in its analytic phases, and deals largely with specializations, the latter is mostly concerned with analysis and systematic general methods. This produces a sharp discontinuity between elementary and "higher" mathematics. While the older mathematics does not see the woods for the trees, the newer is so concerned with vast forests that the trees almost cease to exist. The way out of this dilemma is to master the general theory by repeated study of the details of special cases. This is our author's didactic platform. In this book the general is usually preceded by the specific. Lengthy considerations of a general nature are often relegated to the supplements which follow each chapter. These also contain much additional material, often of the greatest interest, which may be skipped at a first reading.

The content of the book is rich and full of interest. The concepts of real number, function, limit and continuity are treated in Chapter I. Chapter II

opens with the definite integral, treated first as an area, then analytically as the limit of a sum with computed examples. Then comes the derivative, introduced by the problem of tangency before the analytic definition is given. The derivative as a velocity and general rules for differentiation follow. Next the integral is considered as a function of its upper limit and its derivative computed. Thence the central relationship of the calculus. The chapter also contains the mean value theorems for derivatives and integrals, simple methods of graphical integration, and excellent general remarks on the connection between the mathematical integral and derivative and their counterparts in the natural sciences. The supplement deals with the existence of an integral of a continuous function and the connection between theorems of mean value. Chapters III and IV now develop systematic methods for computing derivatives and integrals and Chapter V is concerned with applications to geometery and mechanics.

The following chapters turn to more specific problems: Taylor's formula and polynomial approximations for functions (VI), numerical computation, including Stirling's formula (VII), infinite series and other limiting processes (VIII), Fourier's series and trigonometric interpolation (IX), and finally the differential equations of the simplest oscillatory phenomena (X).

The treatment is uniformly elegant and clear, the exposition growing more concise toward the end of the book. The most interesting chapter, perhaps, is the one on Fourier's series. The development of functions $f(x)$, such that $f(x)$, $f'(x)$, $f''(x)$ are continuous in a finite number of subintervals of $(-\pi, \pi)$ and which approach definite limits as the boundary points are approached from either side, is treated in detail. It is hard to see how the treatment could be improved.

LOUIS BRAND

*Fourfold Geometry.* By D. B. Mair. D. Van Nostrand Co. viii+183 pp.

This book deals with the geometry of a four-dimensional Euclidean space and the equivalent special theory of relativity. Vector methods and notation are used systematically. The terminology employed seems excessively bizarre in some instances; such terms as "biplane" and "backhand stroke" with their respective aviation and tennis connotations are rather distracting to the mathematical reader. On the whole, though, the exposition is quite clear. The printing and page arrangement are agreeable to the eye.

JESSE DOUGLAS

# MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, University of Texas, Austin, Texas.

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## CLUB TOPICS

### 1929 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following list of important 1929 centennial dates is presented. They may be of interest and value to mathematics teachers as suggestions for program topics or special investigations and reports. Additional details may be found in the standard histories of mathematics.

The last four items in the list are of particular interest as one contemplates the significance of Jacobi's elliptic functions, Sturm's theorem, Lobachevsky's non-Euclidean geometry, and Bowditch's translation of Laplace and the resulting mathematical development in the one-hundred year period since their publication.

529. Edict of Emperor Justinian closing the schools at Athens, marking the close of the long period of approximately one thousand years of mathematical development in Greece.

929. Death at Bagdad of Albategni (or Albategnius), important contributor to development of trigonometry, and one of the first of the many Arabian astronomers.

c. 1029. Death of Al-Kharki, last of the real contributors to mathematics at Bagdad, author of an algebra, the *Fakhri*, which "ranks as the most scholarly algebra of the Arabs" (D. E. Smith).

1629. Birth of Huygens, Dutch physicist and mathematician, one of the greatest scientists of the seventeenth century, who solved important problems connected with the cissoid, tractrix, catenary, cycloid, logarithmic curve, evolutes, etc.

1629. First use of brackets by the Flemish mathematician, Girard. (W. W. R. Ball). According to U. G. Mitchell (J. W. Young: *Fundamental Concepts of Algebra and Geometry*, p. 237), Girard also introduced parentheses and signs for square and cube root the same year.

1729. Publication of *Arithmetick: Vulgar and Decimal*, written by Professor Isaac Greenwood of Harvard College, the first arithmetic written by an American author and printed in America. Three copies are in existence, two in the Harvard Library and one in Library of Congress.

1829. Birth of Moritz Cantor, author of *Vorlesungen über Geschichte der*

[1] This Monthly, vol. 35 (1928), p. 437 for a list of 1928 events and for references to previous volumes for corresponding lists for 1925, 1926, and 1927. The reference there given to vol. 33 (1927) p. 141, is in error in the volume number. It should read vol. 34 (1927), p. 141.

*Mathematik*, foremost writer and authority of the nineteenth century on the history of mathematics.

1829. Death at the age of 26 of Abel, Norwegian mathematician, who gave the first proof of the impossibility of the algebraic solution of the quintic equation, and who contributed to the study of elliptic and Abelian functions.

1829. Death of Thomas Young, eminent British mathematical physicist and physician, whose researches paved the way for the acceptance of the undulatory theory of light.

1829. Publication of Jacobi's treatise on *Elliptic Functions*, containing the theory and modern notation of the study of elliptic functions.

1829. Publication of Sturm's theorem, the celebrated method for determining the number and location of the real roots of an algebraic equation.

1829. Publication in the Kasan *Messenger* of a paper by Lobachevsky on "New elements on geometry, with a complete theory of parallels," based upon the denial of the Euclidean postulate of parallel lines. First publication of the foundations of non-Euclidean geometry.

1829. Beginning of publication of Bowditch's translation of Laplace's *Mécanique Céleste*, which gave powerful stimulus to the advancement of mathematical learning in the United States and which, according to Edward Everett, "may be considered as opening a new era in the history of American science."

## CLUB ACTIVITIES

### *The Newtonian Society of Lehigh University, Bethlehem, Pa.*

The Newtonian Society of Lehigh University, organized October 9th, 1927, presented to its members a varied program during the year 1927–1928. The society is conducted by the freshmen with the assistance of a faculty adviser who is a member of the department of mathematics.

The purpose of the society is to promote interest in mathematics among certain members of the freshman class, to give its members an opportunity for intellectual activity outside the class room and to promote friendship between its members, and between its members and the members of the faculty.

Membership in the Newtonian Society is an honor conferred only upon students of the freshman class who have attained high standing and distinction in mathematics. The membership is limited to forty.

The officers for the first semester of the year 1927–1928 were: P. M. Judd, president; L. M. Sobo, vice-president, treasurer; S. Sinick, secretary; Professor Tomlinson Fort, faculty adviser. For the second semester, the officers were: S. Sinick, president; E. F. Underwood, vice-president, treasurer; W. J. Tomlinson, secretary; Professor Tomlinson Fort, faculty adviser.

The officers for the first semester of the year 1928–1929 are: E. B. Douglas, president; C. C. Seabrook, vice-president, treasurer; J. W. Schneider, secretary; W. Altman, press agent; Professor Frank M. Weida, faculty adviser.

The programs presented to the Society included the following papers:

Oct. 26, 1927. "*A* versus *B*," by Professor F. M. Weida.

Nov. 6, 1927. "Some exercises in mental arithmetic," by S. Sinick; "Troublesome numbers," by L. M. Sobo; "Repeating decimals," by L. C. Martin.

Nov. 30, 1927. "The slide-rule and computing machines," by Professor F. M. Weida.

Dec. 14, 1927. "Life of Newton," by E. F. Underwood; "The binomial theorem," by G. L. Frantz; "The law of gravitation," by E. E. Wychoff.

Jan. 11, 1928. "Zero," by L. M. Sobo; "The Greeks and mathematics," by M. G. Tunick.
Feb. 22, 1928. "Greek mathematicians," by M. G. Tunick; "The Pythagorean theorem," by C. R. Lowenstein; "Archimedes," by E. E. Wychoff; Cello selection, by Professor K. W. Lamson.
Mar. 7, 1928. "A problem," by W. J. Tomlinson; "Magic squares," by Max Carlin; "Trisection of an angle, squaring the circle, duplication of the cube," by W. Forstall, Jr.
Mar. 21, 1928. "Comets and their paths," by Professor J. H. Ogburn; "The nine-point circle," by C. Kolodin; "Mathematics of artillery," by F. B. Freese.
April 18, 1928 "Computation of logarithms," by Professor L. L. Smail; "Napier and Briggs" by A W. Thornton; "A tetrahedron problem," by W. J. Tomlinson.
May 2, 1928. "Comparison of school life in America with that in Asia Minor," by Mr. Lagouros; "Time systems," by Professor J. E. Stocker.
May 16, 1928. "Pi Mu Epsilon," by Professor Tomlinson Fort.

(Report by W. J. Tomlinson, Secretary)

### *The Mathematics Club of Oklahoma University, Norman, Oklahoma*

The officers for the year were: Eddie Eaves, president; Miss Shaul, vice-president; Lawrence Montgomery, secretary and treasurer.

The program for the year was:

October 20, 1927. "Interesting biographies of mathematics," by Dr. Court.
November 3, 1927. "Contributions of the Greeks to the calcalus," by Max Heazlet.
November 17, 1927. "Squaring the circle," by Eunice Lewis; "Trisection of an angle," by Dr. Court.
December 8, 1927. "Poincaré's non-Euclidean world," by Mr. Camp.
December 15, 1927. A Christmas party with mathematical games, and refreshments.
January 12, 1928. "Any figure which can be constructed with ruler and compasses can be constructed with compasses alone," by Eddie Eaves.
February 16, 1928. "Aesthetics of mathematics," by Marjorie Dunn.
May 3, 1928. The mathematics club reorganized under the name of Mu Alpha Theta.

The officers were: Lawrence Montgomery, president; Eddie Eaves, vice-president; Maybelle Ausherman, secretary; Elizabeth Womack, treasurer; and Max Heazlet, librarian.

May 10, 1928. A business meeting to continue organizing Mu Alpha Theta.
Oct. 4, 1928. Election of 16 new candidates who had become eligible for Mu Alpha Theta.
Oct. 25, 1928. Initiation of new members, and a talk on "Curious numbers," by Mr. Hampton.

(Report by Maybelle Ausherman, Secretary)

### *Report of the Mathematics Club of Northwestern University*

The club was organized on January 10, 1916.

The officers for the year 1927–1928 were: Violet M. Andrews, president; John O. Chellevold, vice-president; B. Evelyn Frank, secretary; Pearl Bierman, treasurer; Dr. Lois Griffith, faculty adviser.

The program during that time was:

Nov. 16, 1927. "Maxima and minima," by Professor D. R. Curtiss.
Dec. 1, 1927. "Synthetic discussion of conic sections," by Miss Violet M. Andrews.
Dec. 15, 1927. "Simple inequalities relative to a circle," by Professor H. Simmons.
Jan. 19, 1928. "Pythagorean triads," by Mr. Edwin Comfort.
Feb. 16, 1928. "Approximation to power series by means of continued fractions," by Dr. H. Wall.
Mar. 15, 1928. "Types of indeterminate equations," by Miss Elizabeth Williams.
Mar. 29, 1928. "Consequences," by Mr. Albert May.
April 19, 1928. "Hyperbolic functions," by Mr. John O. Chellevold.
May 3, 1928. "A particular solution of a three-body problem," by Mr. H. Glenn Peebles.
May 24, 1928. The annual picnic was held in the Cook County Forest Reserve.

(Report by Albert E. May)

### *The Mathematics Club of Smith College, Northampton, Massachusetts*

The programs of the Mathematics Club of Smith College for the year 1927–1928 were:

Oct. 26, 1927. Short business meeting. Officers for 1927–1928: Evelyn Nieman, president; Evelyn Puffer, secretary; Victoria Pederson, treasurer; Faith Curtis, historian.

Nov. 14, 1927. Joint meeting with the Physics Club. "Sound," by Ethel Hutchberger.

Dec. 13, 1927. Christmas Party. Roll call—each member responded with a mathematical formula. Play—"The Eternal Triangle," by Edith Cook, Marguerite Kirk, Elizabeth Rowbatham, and Marjorie G. Banks. Games. Christmas carols.

Jan. 16, 1928. "Mathematical instruments": The sundial, by Isabelle Williams; Calculating machines, by Margaret Chittim; The history and use of the slide rule, by Ruth Sumner and Elise Thomas.

Feb. 20, 1928. "History of series," by Rachel Howe.

Mar. 12, 1928. "The use of differential equations in physics," by Frances Gilbert; "Primitive concepts of numbers," by Ruth Puffer.

April 16, 1928. "Welcome to new members"; "Application of the theory of least squares to statistics," by Adele Hamerschlag; "Mathematics in physical chemistry," by Mary L. Peterman.

May 7, 1928. Election of officers for 1928–1929: Sally Hill, president; Marjorie G. Banks, secretary; Edith Cook, treasurer; Louise Spetnagel, Historian. "Opportunities for girls majoring in Mathematics," by Miss Knapp of the Personnel Department; "Einstein's theory of relativity," by Sally Hill.

May 28, 1928. Party. "Dead of Night" pantomine. New members. Roll call: each member responded with a name of a person famous in the mathematical field whose last name began with the same letter as the member's. Games.

(Report by Marjorie G. Banks)

### *The Mathematics Club of the University of Nebraska, Lincoln, Nebraska*

Meetings held during the year 1927–1928 were as follows:

Nov. 10, 1927. "Making and using tables of logarithms to ten or more decimal places," by Professor C. C. Camp; business meeting at which the following officers were elected for the first semester: Joe Styskal, president; Frank Roth, vice-president; H. Herbert Howe, secretary and treasurer.

Dec. 8, 1927. "A machine for solving $n$th degree equations," by Merrill Flood; New members initiated; Short business meeting.

Jan. 12, 1928. "Flatland," by Druscilla Winchester; "Madame Sophia Kowalewski, a Russian mathematician," by Alice Bromwell; "Iteration," by Gladys Baldwin; Short business meeting.

Feb. 9, 1928. "Mathematics of insurance," by Mr. Hiller; Business meeting, at which the following officers were elected for the second semester: Theodore Jorgensen, Jr. president; Druscilla Winchester, vice-president; Wm. Louis Bitney, secretary and treasurer; Professor C. C. Camp, faculty adviser.

Mar. 8, 1928. "History of pi," by H. Herbert Howe; "Life and work of Archimedes," by Dorothy Swanson; "Trisection of an angle," by G. E. Evans. A business meeting followed.

April 5, 1928. Open meeting. "Mathematics of astronomy," by Professor Swezy.

May 17, 1928. Banquet and Installation of a chapter of Pi Mu Epsilon. The following officers were elected: Theodore Jorgensen, Jr., director; Evelyn Fate, vice-director; G. E. Evans, secretary; Frank Roth, treasurer; Wilma G. Worden, librarian; Professor C. C. Camp, secretary and faculty adviser.

May 24, 1928. Picnic.

(Report by Paul Bartunek, Secretary)

### *The Shuttleworth*[1] *Mathematical Society of the University of Saskatchewan, Saskatoon, Saskatchewan, Canada*

Officers for the year: Dean G. H. Ling, honorary president; G. S. Ling, president; R. Meldrum, vice-president; Marian Evans, secretary-treasurer; Jean Foote, recording secretary; A. H. Thiesse junior representative; Miss Olive Hughes, faculty adviser.

The following is the program of the Shuttleworth Mathematical Society for the year 1927–28:

Nov. 10, 1927. "Theory of determinants," by G. S. Ling; "Compound and simple interest," by Mary Sherrick.

Dec. 1, 1927. "On the surface of the sphere," by Jean Foote; "The equation of the fifth degree," by A. H. Thiessen; A letter from H. H. Ferns, University of Illinois, formerly a member of the Society.

Jan. 26, 1928. "Contribution of mathematics to life," by A. McCauley; "The pendulum problem," by O. Gratias; A letter from K. Folley, of the University of Toronto, formerly a member of the Society.

Feb. 9, 1928. Annual dinner. Speaker, Professor A. J. Pyke.

Mar. 1, 1928. "Irrational numbers," by Marian Evans; "An engineering problem involving maxima and minima," by J. Sexton; A letter from C. D. Clark of Oxford University, formerly a member of the Society.

Mar. 29, 1928. "The compound interest law," by R. Meldrum; "A problem concerning the earth's motion around the sun," by J. M. Muth.

(Reported by Marian A. Evans, Secretary)

### *The Mathematical and Physical Society of the University of Toronto*

Since the centenary celebrations of this University were being held at this time, it was decided by the executive to have the associated divisions of science dealt with from their earliest history till the year 1827; these to be prepared by undergraduate members. In addition, papers were read by members of the faculty.

The program for the year 1927–28 was:

Oct. 20, 1927. "The international congress in physics held in Italy Sept. 11–22, 1927," by Dr. J. C. McLennan.

Nov. 3, 1927. "Heat up to 1827," by Mr. J. D. Milne; "The wearing of spectacles," by Dr. McTaggart.

Nov. 17, 1927. Open House Debate. "Resolved that theories have been more detrimental than useful to the advancement of science." Affirmative: Mr. A. Cameron, '28 and Miss D. Quance '31; Negative: Mr. W. D. Scholfield, '28 and Miss A. Watson, '31.

Nov. 28, 1927. Annual Dance.

Dec. 1, 1927. "Mathematics up to 1827," by Mr. A. Tucker, '28; "Geometrical constructions," by Dr. Beatty.

Dec. 15, 1927. "Radio in 2027," by Mr. F. M. E. Holmes, '30; Social meeting.

Jan. 5, 1928. "Light up to 1827," by Mr. L. Leppard, '30; "Paper in physics," by Dr. Burton.

Jan. 19, 1928. "Electricity & Magnetism to 1827," by Miss E. Allin; "Recent developments with cathode rays," by Mr. J. C. Ireton.

Feb. 2, 1928. "Astronomy to 1827," by Mr. J. Archibald, '28; "The Dominion Observatory, Ottawa," by Miss W. Woollcombe, '28; "The Dominion Astro-Physical Observatory, Victoria," by Mr. P. Millma, '29.

Feb. 16, 1928. "Chemistry up to 1827," by Mr. R. McKay, '29; "Living machinery," by Dr. Satterly.

Mar. 1, 1928. Open Meeting. Nominations for Executive for the year 1928–29.

Mar. 15, 1928. Annual Meeting. Election of Officers. Executive for the year 1927–28: Honorary president, Professor E. F. Burton, B. A., Ph.D.; Miss M. Annetts, '28, president;

[1] The name is a memorial to Roy Shuttleworth, first president of the Society, who gave his life in the War.

Mr. D. B. DeLury, '29, vice-president; Miss W. D. Woollcombe, '28, corresponding secretary; Miss M. Sutherland, '29, recording secretary; Mr. F. M. E. Holmes, '30, treasurer. Representatives: Miss E. J. Allin, graduate; Mr. J. C. Archibald, 4th year; Mr. J. E. Smart, 3rd year; Mr. L. B. Leppard, 2nd year; Miss A. Watson, 1st year; Mr. J. S. Ball.

### *Gamma Eta Mu, University of Redlands, Redlands, California*

Gamma Eta Mu Club of the University of Redlands held monthly meetings during the year 1927–1928. The programs were as follows:

Oct. 10, 1927. Picnic and business meeting.

Nov. 14, 1927. The program consisted of a talk by Professor Jones in which he discussed the Stefan-Boltzman law of radiation.

Dec. 12, 1927. A problem in calculus was demonstrated; and there was also a discussion of problems pertaining to strength of materials.

Jan. 9, 1928. The program consisted of the presentation of the biographies of men who have done much to develop the mathematical field.

Feb. 6, 1928. There was an illustrated lecture on "Telephotography" by a representative from the Bell Telephone Company.

Mar. 5, 1928. At this meeting two papers, "The teaching of mathematics for citizenship" and "The principles of the Phonodiek" were read.

April 8, 1928. There was a discussion of a problem in physics and a demonstration of a method used in the trisection of an angle.

May 21, 1928. There was a picnic and business meeting. The officers for this year are: Homer Stavely, president; Mildred Smith, vice-president; Madelyn Traviss, treasurer.

(Report by Madelyn Travis, Secretary)

### *The Mathematics Club of Wellesley College, Wellesley, Mass.*

The program of the mathematics Club of Wellesley College for the year 1927–28 was the following:

Oct. 21, 1927. An account of the summer meeting of the American Mathematical Society, by Professor Mary C. Graustein.

Nov. 18, 1927. Talks by club members on magics squares, Greek cross problems, tangrams a n mazes.

Jan. 20, 1928. Admission of new members from the sophomore class, and a lecture on "Linkages" by Professor Raymond K. Morley of Worcester Polytechnic Institute.

Mar. 2, 1928. Talks by club members on skeleton divisions, long division in the Middle Ages, Chinese number systems, number systems on bases other than ten, codes and ciphers.

April 27, 1928. Mathematical play—Flatland.

May 25, 1928. Annual dinner and election of officers for the year 1928–29.

(Report by Doris J. Raine, Secretary)

### *The Mathematics Club of the University of Colorado, Boulder, Colo.*

The following programs were given at the meetings of the club during the year 1927–1928:

Jan. 12, 1928. "Our perpetual calendar," by Walter K. Nelson, Associate Professor of Engineering Mathematics.

Feb. 9, 1928. "A review of the book *Flatland*," by Lucy Cramer.

Mar. 14, 1928. "Quadratics," by Mr. Hazard.

Mar. 28, 1928. A "taffy pull" was held in the Woman's Building. Mathematical puzzles and recreations furnished the amusement.

April 12, 1928. "Non-uniform convergence," by Earle Rainville.

April 26, 1928. "History of logarithms," by Sidney Hacker.

May 10, 1928. "Graphical solutions of equations," by Maybelle Ratliff. Officers for the coming year, 1928–1929, are: Sidney Hacker, president; Earle Rainville, vice-president; and Dorothy Huffman, secretary-treasurer.

(Reported by Dorothy Huffman, Secretary)

# PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

*Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.*

## PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3361. *Proposed by B. F. Finkel, Drury College.*

Find the envelope of a system of circles having for diameter a secant of constant length, $2r$, of a conic.

3362. *Proposed by R. E. Gaines, University of Richmond.*

A triangle $ABC$ circumscribes an ellipse of axes $2a$ and $2b$ so that $A$ lies on one directrix and $B$ on the other, while $AB$, $BC$, $CA$ touch the ellipse at $P$, $Q$, $R$. Show that the envelope of $QR$ and the locus of $C$ are ellipses with $2a$ as the major axis of one and the minor axis of the other, while $2b$ is the mean proportional between the other axes.

3363. *Proposed by Otto Dunkel, Washington University.*

In an urn there are $k+1$ counters of which one is a blank while the others are numbered from 1 to $k$. A single counter is withdrawn and then replaced, and this is continued until there are $n$ such drawings. In how many ways may the $n$ drawings be made so that in the first $r$ or more drawings only blanks are obtained and in the rest of the $n$ drawings no numbered counter is followed by as many as $r$ consecutive blanks? Also determine in how many ways $n$ drawings may be made so that $r$ or more consecutive blanks are drawn.

3364. *Proposed by Otto Dunkel, Washington University.*

In an urn there are $k+1$ counters of which one is a blank while the others are numbered from 1 to $k$. A single counter is withdrawn and then replaced and this is continued until there are $n$ drawings. In how many ways may the $n$ drawings be made so that in the first $r$ drawings only blanks are obtained and the $(r+1)$th drawing is a number, while in the rest of the $n$ drawings no numbered counter is followed by precisely $r$ blanks. Also determine the number of ways the $n$ drawings may be made so that precisely $r$ blanks are drawn in succession.

## SOLUTIONS

3162 [3158; 1926, 47]. *Proposed by Otto Dunkel, Washington University.*

Show that the absolute values of the imaginary roots (also of the negative roots when $n$ is even) of the equation

$$x^{n+1} - (k+1)x^n + k = 0, \qquad k > 0, \quad n \geqq 2,$$

lie between $[k/(k+2)]^{1/n}$ and unity; also that the absolute values of the imaginary roots are greater than that of the negative root of (a) the given equation when $n$ is even, of (b) the equation obtained by replacing $n$ by $n-1$ when $n$ is odd.

If $k \geqq 1$ the absolute values of the imaginary roots (and negative roots) lie between .61 and 1.

*Solution by the Proposer.*

The real roots of the given equation will first be examined, and it will be convenient to write

$$(1) \qquad \begin{aligned} f(x) &= x^{n+1} - (k+1)x^n + k, \qquad k > 0, \quad n > 1, \\ f'(x) &= x^{n-1}[(n+1)x - n(k+1)]. \end{aligned}$$

If $n$ is odd there can be no negative roots. If $n$ is even, $f'(x)$ is positive for $x<0$, $f(-1)=-2$, and $f(0)=k>0$; and therefore there is one and only one negative root and it lies between $-1$ and 0.

In all cases $x=1$ is a root, and it will be shown that there is one and only one other positive root, which will be denoted by $a$. Let $\bar{x}=n(k+1)/(n+1)$ so that $f'(\bar{x})=0$; then $\bar{x}<1$, $\bar{x}=1$, or $\bar{x}>1$ according as $k<1/n$, $k=1/n$, or $k>1/n$. If $k<1/n$, then as $x$ takes on the values 0, $\bar{x}$, 1, $f(x)$ takes on values $k$, $-$, 0, and then increases. Hence $a$ and 1 are the only positive roots and $0<a<\bar{x}<1$. If $k=1/n$, then $x=1$ is a double root and there is no other positive root. If $k>1/n$, we find in a similar manner that 1 and $a$ are the only positive roots, and that $1<\bar{x}<a<1+k$.

For the consideration of the imaginary roots it will be convenient to set $x=1/y$, and to examine the equation

$$(2) \qquad ky^{n+1} - (k+1)y + 1 = 0.$$

This equation has the real positive roots 1 and $A=a^{-1}$, and thus $kA^{n+1}-(k+1)A+1=0$. Suppose that $y$ is an imaginary root. Then by subtracting the two equations in $y$ and $A$ and dividing by $y-A$, which is surely not zero, we have

$$(3) \qquad (k+1)/k = y^n + Ay^{n-1} + \cdots + A^{n-1}y + A^n.$$

We shall prove that $|y|>1$, and then it follows that $|x|<1$, where $x$ is an imaginary root of $f(x)=0$. It will be noticed that the proof applies also if $x$ is a negative root. For suppose that $|y| \leqq 1$ and consider first the case $kn \neq 1$. Then from the above we have

$$(4) \qquad \begin{aligned} (k+1)/k &< |y|^n + A|y|^{n-1} + \cdots + A^{n-1}|y| + A^n, \\ &< 1 + A + \cdots + A^{n-1} + A^n. \end{aligned}$$

The equality sign cannot be used in the first inequality since the different powers of $y$ have different angles. Now (3) above is satisfied by $y=1$. Hence the right side of the last inequality in (4) is equal to $(k+1)/k$, and we have the contradiction $(k+1)/k<(k+1)/k$. If $kn=1$, then $A=1$ and the last inequality becomes the contradiction $n+1<n+1$. Hence $|x|<1$, if $x$ is an imaginary or negative root of $f(x)=0$.

We shall now determine lower bounds for the imaginary and negative roots, and for this purpose it will be convenient to set $z=-x$ and to write

$$z^{n+1}+(k+1)z^n=(-1)^n k. \tag{5}$$

Suppose first that $z$ is an imaginary root, then $k<|z|^n[|z|+(k+1)]$, and hence

$$|z|^n>\frac{k}{[|z|+k+1]}>\frac{k}{(k+2)}, \tag{6'}$$

where the last inequality follows since $|z|<1$. Hence we have

$$1>|x|>(k/k+2)^{1/n}. \tag{6}$$

If $x$ is a negative root, $z$ is positive, and the reasoning is altered merely by replacing the first inequality sign in (6') by the equality sign, the second and last inequality signs remaining since $|z|<1$, if $z$ corresponds to a negative root. Hence (6) is true also for the negative root if there is one.

If $n$ is even there is a negative root, say $-b$, so that $z=b$ is a root of (5). Hence $b^n[b+k+1]=k$. Suppose then that $x=-z$ is an imaginary root in this case, and that $|x|=|z|\leqq b$. Then

$$k<|z|^n[|z|+k+1]\leqq b^n[b+k+1]=k.$$

Here we have the contradiction $k<k$, and hence $|x|>b$. If $n$ is odd consider the equation $z^n+(k+1)z^{n-1}=k$ and its root $b'$ where $-b'$ is the negative root of the $x$ equation. Also let $x=-z$ be an imaginary root of (5). Suppose that $|z|\leqq b'$; then as before

$$k<|z|^n[|z|+k+1]\leqq b'[b'^n+(k+1)b'^{n-1}] \quad \text{or} \quad k<b'k.$$

The last inequality follows since the bracket with $b'$ is equal to $k$. But the last inequality is impossible since $b'<1$. Hence the absolute value of any imaginary root of $f(x)=0$ is greater than the absolute value of its negative root if $n$ is even. If $n$ is odd it is greater than the absolute value of the negative root of the corresponding equation in which $n$ is replaced by $n-1$.

It will now be shown that if $k$ is fixed and $n$ increases passing through even values, then $b$ increases. Let $b$ and $\bar{b}$ be the roots corresponding to $n$ and $n+2$. Then $\bar{b}^{n+3}+(k+1)\bar{b}^{n+2}=k$ or $\bar{b}^{n+1}+(k+1)\bar{b}^n=k/\bar{b}^2>k$, since $\bar{b}<1$. But $b^{n+1}+(k+1)b^n=k$, and therefore we must have $\bar{b}>b$. If now $n$ is fixed and even and $b$ increases then $k$ increases. For $k=b^n(b+1)/(1-b^n)$, and as $b$ increases the numerator increases and the denominator decreases. Conversely as $k$ in-

creases $b$ increases. Therefore the absolute values of the imaginary roots are greater than the positive root of $z^3+(k+1)z^2-k=0$, or of $z^2+kz-k=0$. Hence if $x$ is an imaginary root,

$$|x| > \frac{1}{2}[(4k + k^2)^{1/2} - k] > \frac{k}{k+1}.$$

If $k \geqq 1$ the value of the middle term in the above inequalities is greater than 0.6180, and in this case $|x|$ lies between 0.61 and 1.

In the case $k<1/n$ we have $0<a<1$, and in order to complete the discussion it will be shown by the same kind of argument that the absolute values of the imaginary roots are less than $a$. Using the equation (2) with the root $y=1$ removed we may write the equation

$$(k + 1)/k = y^n + y^{n-1} + \cdots + y + 1,$$

which is satisfied by $A=1/a$ and the imaginary roots. Suppose that $y$ is an imaginary root and that $|y| \leqq A$; then

$$\begin{aligned}(k + 1)/k &< |y|^n + |y|^{n-1} + \cdots + |y| + 1,\\ &< A^n + A^{n-1} + \cdots + A + 1 = (k + 1)/k.\end{aligned}$$

Here we have again the absurdity $(k+1)/k<(k+1)/k$, and therefore $|y|>A$. The same reasoning applies to the negative root. Thus if $x$ is an imaginary root of $f(x)=0$, or a negative root, and if $k<1/n$, $|x|<a$.

It may also be shown by arithmetical reasoning that if $n$ is fixed $a$ increases with $k$; if $k$ is fixed $a$ increases with $n$. The equation $f(x)=0$ for $n=4$ has been solved as an illustrative example in this Monthly [1926, 232].

3288 [1927, 491]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The variable line on which two given circles determine chords of equal length envelops a parabola. The midpoint of the line of centers of the circles is the focus of the parabola, and its tangent at the vertex coincides with the radical axis of the two given circles.

I. *Solution by C. S. Carlson, University of Iowa*

Let us choose the line of centers of the two given circles as the axis of $X$ and the perpendicular bisector of the segment joining the centers as the axis of $Y$. Then the equations of the two given circles can be written in the form

$$x^2 + y^2 - 2gx + f = 0 \cdots, \tag{1}$$

$$x^2 + y^2 + 2gx + f' = 0 \cdots. \tag{2}$$

Solving (1) with the line $y=mx+c$ we have the relation

$$x^2(1 + m^2) + x(2mc - 2g) + f + c^2 = 0$$

from which the roots $x_1$ and $x_2$ give the relations

$$x_1 + x_2 = -(2mc - 2g)/(1 + m^2) \tag{3}$$

$$x_1x_2 = (f + c^2)/(1 + m^2). \tag{4}$$

Squaring (3) and subtracting four times (4) we find

$$x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 = 4(-m^2f - 2mcg + g^2 - f - c^2)(1 + m^2)^{-2}. \tag{5}$$

Now (5) gives the square of the projection of the chord on the axis of $X$.

Using the second circle, a similar procedure gives the square of the projection of the second chord on the axis of $X$: $4(-m^2f' + 2mcg + g^2 - f' - c^2)(1 + m^2)^{-2}$. Since the chords are equal, by hypothesis, and are projected from the same line upon the axis of $X$, we have

$$-m^2f - 2mcg + g^2 - f - c^2 = -m^2f' + 2mcg + g^2 - f' - c^2;$$

whence $m^2(f' - f) - 4mgc + f' - f = 0$. Solving this relation for $c$ and substituting in $y = mx + c$, the equation of the variable line now becomes $y = m(x - a) - am^{-1}$ when $a = (f - f')/4g$. From analytic geometry (See C. Smith, *Conic Sections*, paragraph 94, page 112) we know that this line is always tangent to the parabola $y^2 = -4a(x - a)$. From this equation it is now evident that the radical axis of the two circles is the tangent at the vertex and that the focus is the mid-point of the line of centers.

The first part of this problem is found in C. Smith's *Conic Sections* (1910 edition), page 134.

Also solved by Theodore Bennett, Rufus Crane, A. Pelletier, and Paul Wernicke.

II. *Solution by Otto Dunkel, Washington University.*

The envelope may be determined geometrically. Let $C_1$, $C_2$ be the centers of the two circles, and $F$ the mid-point of $C_1C_2$. Let a straight line determine the chords of the two circles $A_1B_1$, $A_2B_2$ which are of the equal lengths $2c$. Let the projections of $C_1$, $F$, $C_2$ on this secant be $M_1$, $N$, $M_2$; then $NM_1$ and $NM_2$ are of equal lengths $m$. Hence for either circle $NA \cdot NB = (m - c)(m + c)$, and thus the tangents to the circles from $N$ are of equal length, or, if $N$ lies within a circle, the product of the segments of the chords through $N$ is the same for both circles. Hence $N$ lies on the radical axis of the two circles, which cuts $C_1C_2$ perpendicularly in $V$, and also $FN$ is parallel to $C_1M_1$ and $C_2M_2$ and therefore perpendicular to $M_1M_2$. This determines the construction for the secant $M_1M_2$. Obviously the points may be lettered so that $NA_1$ and $NA_2$ are equal in length, and the same thing is then true of $NB_1$ and $NB_2$.

Consider two such secants $NM_1$ and $N'M_1'$ intersecting in $P'$, where $FNM_1$ and $FN'M_1'$ are right angles. A circle may be passed through $F$, $N$, $N'$, $P'$ with center at the mid-point of $FP'$. As $N'$ approaches $N$, the point $P'$ approaches a limiting position $P$ on $NM_1$ and the circle approaches a circle through $F$ and $P$ tangent to $VN$ at $N$. To construct $P$ we draw the fixed auxiliary line $DD'$ perpendicular to $VF$ at $D$ such that $V$ is the mid-point of

$DF$. Draw any line $FNF'$ cutting $VN$ and $DD'$ in $N$ and $F'$ respectively. Through $F'$ draw the line $F'P$ perpendicular to $DD'$ at $F'$, and $NP$ perpendicular to $FF'$ at $N$ meeting the first perpendicular in $P$. Draw $FP$ cutting in $C$ the perpendicular $NC$ to $VN$ at $N$. Then $FC$, $CP$, $CN$ are of equal lengths, and $C$ is the center of the limit circle and $P$ is the limit point on $NM_1$, and this line is the tangent to the envelope at $P$. Since $FN = NF'$, $FP = F'P$; hence the point $P$ lies on a parabola with focus at $F$, tangent to $VN$ at $V$, and having $DD'$ as directrix.

3292 [1927, 491]. *Proposed by R. E. Gaines, University of Richmond.*

A chord of a given conic is of constant length; find the locus of its pole.

*Solution by the Proposer.*

The equation of the conic may be written

$$ax^2 + by^2 + 2gx = 0, \qquad bg \neq 0; \tag{1}$$

and, if $(\alpha, \beta)$ is the pole of the chord, the equation of the chord is

$$(a\alpha + g)x + b\beta y + g\alpha = 0. \tag{2}$$

If the length of the chord is $2l$ and if its extremities have the coördinates $(x', y')$, $(x'', y'')$, then

$$(x'' - x')^2 + (y'' - y')^2 = 4l^2. \tag{3}$$

Since the coördinates of the extremities satisfy (2) we may write (3) in the form

$$[(x'' + x')^2 - 4x'x''][1 + (a\alpha + g)^2 b^{-2}\beta^{-2}] = 4l^2. \tag{4}$$

Eliminating $y$ from (1) and (2) we have

$$[ab\beta^2 + (a\alpha + g)^2]x^2 + 2g[a\alpha^2 + b\beta^2 + g\alpha]x + g^2\alpha^2 = 0 \tag{5}$$

as the equation for $x'$ and $x''$. Inserting the sum and the product of the roots of this equation in (4), and replacing $(\alpha, \beta)$ by $(x, y)$, we obtain the equation of the locus required:

$$g^2[ax^2 + by^2 + 2gx][(ax + g)^2 + b^2y^2] = bl^2[(ax + g)^2 + aby^2]^2.$$

If $a \neq 0$, the conic has the center $(-g/a, 0)$, and the locus possesses this point as an isolated double point.

Also solved by William Hoover, W. J. Patterson, Paul Wernicke and Roscoe Woods.

3299 [1927, 538]. *Proposed by H. W. Reddick, Cooper Institute of Technology, New York City.*

Solve the differential equation $(d^{n+1}y/dx^{n+1}) - (dy/dx) - (ny/x) = 0$, (a generalization of Problem 3227).

*Solution by R. H. Sciobereti, Berkeley, California.*

In the form

$$xy^{(n+1)} - xy' - ny = 0, \tag{1}$$

the proposed differential equation is a particular case of Laplace's equation and its solution may be expressed by means of a definite integral. Assuming $y = \int_C Ze^{xz}dz$ taken along a contour which will be defined later, where $Z$ is an unknown function of $z$, let us differentiate the required number of times with respect to the parameter $x$ and substitute in the original equation; then we shall obtain

$$\int_C Ze^{xz}(P + Qx)dz = 0, \quad \text{where} \quad P = -n \quad \text{and} \quad Q = z(z^n - 1).$$

If $ZQ$ be determined by the condition $d(ZQ) = ZPdz$, or $Z = Q^{-1}e^{\nu}$, where $\nu = \int PQ^{-1}dz$, the definite integral becomes equal to the variation of the function

$$U = ZQe^{xz} = e^{xz+\nu} = e^{xz}z^n/(z^n - 1)$$

along the path of integration $C$. In order to obtain a solution of (1) it will be sufficient to choose the path of integration in such a way that the function $U$ should return to its initial value after the variable $z$ describes the whole contour, provided however that $y = \int_C Ze^{xz}dz$ has a finite value different from 0. The two functions $U$ and $Z$ have the same roots and poles, namely the roots of the polynomial $Q = z(z^n - 1)$. Let us take for the path of integration $C$ a circle whose center will be successively each of the poles $e^{2k\pi i/n} = \alpha_k$ of the function $Z$, then the residue of $e^{xz}z^{n-1}(z^n - 1)^{-2}$ relative to each one of these poles will be a particular solution of the differential equation. The residue corresponding to the pole $\alpha_k$ is obviously the coefficient of $(z - \alpha_k)$ in the development of

$$f(z) = e^{xz}z^{n-1}(z - \alpha_k)^2(z^n - 1)^{-2}$$

about $\alpha_k$. It is therefore $[df(z)/dz]_{z=\alpha_k}$; from Cauchy's fundamental formula $2\pi i f'(x) = \int_C f(z)dx/(z - x)^2$, the residues may be readily calculated as follows: denoting by $\alpha_k$ any one of the roots of the equation $z^n - 1 = 0$, and by $\phi(z)$ the product $\prod_i (z - \alpha_i)(i = 0, 1, 2, \cdots, n-1, \text{but } i \neq k)$ which is equal to the function $(z^n - 1)/(z - \alpha_k)$, we shall have

$$2\pi i \frac{d}{dz}\left[\frac{e^{xz}z^{n-1}}{\phi^2(z)}\right]_{z=\alpha_k} = \int_C \frac{e^{xz}z^{n-1}/\phi^2(z)}{(z - \alpha_k)^2}dz,$$

and after performing all the indicated algebraic operations, the residue $nxe^{\mu}$, where $\mu = \alpha_k x$, relative to the pole $\alpha_k$ is a particular solution of the proposed equation; and since the $n$ particular solutions corresponding to the $n$ roots of unity are linearly independent, it follows that $\sum_k C_k xe^{\mu}$ is also a solution. The general solution may now be obtained by $(n+1)$ quadratures, since $n$ substitutions of the form $y = xe^{\mu}v$, $u = dv/dx$ will reduce the original equation to an

equation of the first order which is integrable by a quadrature. But another particular solution may be obtained by selecting a new and suitable contour of integration. In general, if $x$ is a complex quantity of the form $re^{i\alpha}$ and $z=\rho e^{i\theta}$, the path of integration may be taken as a straight line starting from the origin, not passing through any pole $e^{2k\pi i/n}$ and making with the real axis an angle such that the real part of the product $xz=r\rho e^{i(\theta+\alpha)}$ should be negative, since the function $U=e^{xz}z^n/(z^n-1)$ is to vanish at both extremities. In particular if $x$ is real the definite integral $\int e^{xz}z^{n-1}(z^n-1)^{-2}dz$ will be a solution, when taken along the straight line $\phi=(2k+1)\pi/n$, provided $n/2<2k+1<3n/2$, for $x>0$; and $-n/2<(2k+1)<n/2$, for $x<0$. Hence there is a particular solution

$$C_{n+1}\int_0^\infty e^{\sigma}\rho^{n-1}(\rho^n+1)^{-2}d\rho, \quad \text{where} \quad \sigma = x\rho e^{(2k+1)\pi i/n}.$$

It takes the form

$$C_{n+1}\int_0^\infty e^{-x\rho}\rho^{n-1}(\rho^n+1)^{-2}d\rho,$$

for $x>0$ and $n$ odd.

3300 [1927, 538]. *Proposed by Emma M. Gibson, Central High School, Springfield, Missouri.*

If $u$ and $v$ are particular integrals of the equation

$$(d^3y/dx^3)+P(d^2y/dx^2)+Q(dy/dx)+Ry=0,$$

prove that $y=Au+Bv$ is the complete solution, where

$$A'/v=-B'/u=ce^{\mu}/(u'v-uv')^2 \qquad \left(\mu=-\int Pdx\right),$$

$c$ being a constant and the accents denoting differentiation with respect to $x$. (University of Edinburgh Examinations, 1903.)

*Solution by Frederic H. Miller, Cooper Union Institute of Technology*

Since $u$ and $v$ are two solutions, $y=au+bv$ is a solution, where $a$ and $b$ are constants. Consequently we assume the complete solution in the form $y=Au+Bv$, where now $A$ and $B$ are functions of $x$. Then $y'=A'u+Au'+B'v+Bv'$, and if we make the parameters $A$ and $B$ satisfy the additional relation

$$A'u+B'v=0, \tag{1}$$

we have

$$y'=Au'+Bv',$$
$$y''=A'u'+Au''+B'v'+Bv'',$$
$$y'''=A''u'+2A'u''+Au'''+B''v'+2B'v''+Bv'''.$$

Substitution of these values into the differential equation gives us

$$A''u' + 2A'u'' + B''v' + 2B'v'' + P(A'u' + B'v') = 0. \tag{2}$$

Now from equation (1),

$$B' = -A'u/v, \quad B'' = -(A''uv + A'u'v - A'uv')/v^2;$$

and these values when put into (2) yield

$$A''v(u'v - uv') + A'[2v(u''v - uv'') + (Pv - v')(u'v - uv')] = 0.$$

This is an equation of the first order in $A'$; its solution is

$$A'/v = ce^{\mu}/(u'v - uv')^2, \quad \left(\mu = -\int P dx\right).$$

Combining this with equation (1) we get the desired relation

$$A'/v = -B'/u = ce^{\mu}/(u'v - uv')^2,$$

and hence the complete solution of the differential equation is

$$y = c\left[u\int \frac{ve^{\mu}}{(u'v - uv')^2}dx - v\int \frac{ue^{\mu}}{(u'v - uv')^2}dx\right].$$

3302 [1928, 41]. *Proposed by R. E. Gaines, University of Richmond.*

Determine the envelope of the chord of a central conic which subtends a right angle at a given focus.

*Solution and Generalization by R. Goormaghtigh, La Louvière, Belgium.*

Let $A$ and $B$ be two moving points on two plane curves $(A)$ and $(B)$, the angle subtended by $AB$ at a given point $O$ being a constant $\alpha$, and denote by $E_\alpha$ the envelope of $AB$. If $C$ is a fixed circle with center $O$, the angle between the polars of $A$ and $B$, for the circle $C$, is $\pi-\alpha$. Therefore, $E_\alpha$ is the polar curve, with respect to $C$, of the isoptic curve $\Gamma$, for the angle $\pi-\alpha$, of the polar curves $(A')$ and $(B')$ of $(A)$ and $(B)$ with respect to $C$.

When $(A)$ and $(B)$ are two conics having a common focus $O$ and the same corresponding directrix, $(A')$, $(B')$ are two concentric circles, and $E_\alpha$ is also a conic having the same focus and corresponding directrix as $(A)$ and $(B)$. In Problem 3302, $(A)\equiv(B)$; in this case, the radius of $\Gamma$ being the radius of $(A')$ multiplied by $\sqrt{2}$, the eccentricity of $E_{\pi/2}$ is the eccentricity of $(A)$ divided by $\sqrt{2}$. When $(A)$ and $(B)$ are the same central conic $\Sigma$, and $O$ is any fixed point, the polar curve of $\Sigma$ is a conic and $\Gamma$ is a spiral curve of Perseus when $\alpha\neq\pi/2$, and a circle when $\alpha=\pi/2$. Hence the envelope is the polar curve of the spiral curve of Perseus when $\alpha\neq\pi/2$, and a circle when $\alpha=\pi/2$.

When $O$ is a point on the conic $\Sigma$ and when $\alpha\neq\pi/2$, the polar curve of $\Sigma$ is a parabola, $\Gamma$ is a conic, and $E_\alpha$ is also a conic; in this case, if $\alpha=\pi/2$, $\Gamma$ is a straight line and $AB$ passes through a fixed point (Fregier's theorem). When $\alpha=\pi/2$ and when $O$ is the center of $\Sigma$, $\Gamma$ is a circle, and $E_{\pi/2}$ is also a circle; if $K$ is the radius of $C$ and $2a$, $2b$ the axes of $\Sigma$, the axes of the polar curve are

$2K^2/a$, $2K^2/b$ and $\Gamma$ is a circle with center $O$ and radius $K^2(a^{-2}+b^{-2})^{1/2}$; therefore the radius of $E_{\pi/2}$ is $ab/(a^2+b^2)^{1/2}$. See Problem 3237 [1927, 98].

Also solved by William Hoover, J. H. Neelley, O. J. Ramler, Roscoe Woods, F. L. Wilmer, and the Proposer.

3305 [1928, 92]. *Proposed by Paul Wernicke, Washington, D. C.*

(a) Prove that $x^2+y^2$, where $x$ and $y$ are integers, cannot be the square of an integer unless $x$ or $y$ is divisible by 3.

(b) Modify the theorem for the case that both $x$ and $y$ are divisible by 3.

(c) Generalize the theorem so as to cover other exponents, thereby proving a part of Fermat's greater theorem.

I. *Solution by R. S. Underwood, Texas Technological College.*

The generalized results of which (a) is a special case [$n=2$, $p=3$ in case (1) below] may be stated thus:

If $x^n+y^n=z^n$ has integral solutions ($n$ an integer),

(1) when $n=(p-1)k(p\geqq 3)$, $x$ or $y$ must be divisible by $p$ (a prime),

(2) when $n=\frac{1}{2}(p-1)k(p\geqq 5)$, $x$, $y$, or $z$ must be divisible by $p$.

Proof of (1): Let $n=(p-1)m$, and let neither $x$ nor $y$ be divisible by $p$. Then, by a well-known theorem, due to Fermat,

$$x^n \equiv y^n \equiv 1 \pmod{p}.$$

When $z$ is not divisible by $p$,

$$z^n \equiv 1 \pmod{p}.$$

When $z$ is divisible by $p$,

$$z^n \equiv 0 \pmod{p}.$$

In either case, $x^n+y^n \equiv 2 \pmod{p} \not\equiv z^n \pmod{p\geqq 3}$,

and the theorem is proved.

Proof of (2): Let $n=\frac{1}{2}(p-1)k$, and let $x$, $y$, and $z$ be prime to $p$. Then, when $p\geqq 5$, $x^n \equiv \pm 1$, $y^n \equiv \pm 1$, $z^n \equiv \pm 1 \pmod{p}$. Therefore

$$x^n + y^n \equiv \pm 2 \text{ or } 0 \pmod{p} \not\equiv z^n \pmod{p};$$

and the theorem is proved.

II. *Solution by Laurence Hampton, University of Oklahoma.*

If $p$ is a prime number and $p^k$ is the highest power of $p$, $k\geqq 0$, which is contained in both $x$ and $y$, then either $x$ or $y$ must contain the factor $p^{k+1}$ if the equation $x^{p-1}+y^{p-1}=z^{p-1}$ is solvable in integers. The proof is as follows: Since $x$ and $y$ are each divisible by $p^k$, $z$ must also be so divisible, and we may take out this factor and write $X^{p-1}=Z^{p-1}-Y^{p-1}$. Suppose $Y$ does not contain $p$ as a factor; then $Y^{p-1}\equiv 1 \pmod{p}$. Also $Z^{p-1}\equiv 0$ or $1 \pmod{p}$ according as $Z$ does or does not contain $p$ as a factor. Hence $X^{p-1}\equiv -1$ or $0 \pmod{p}$. But only the

second of the two is possible. Hence $x$ is divisible by $p^{k+1}$ and $z$ is divisible by $p^k$.

Also solved by A. Pelletier and J. E. Rowe.

3306 [1928, 92]. *Proposed by H. A. Mangan, Vandalia, Mo.*

A man bought a horse for $A$ dollars and agreed to pay interest and principal in $n$ equal monthly installments, interest $r$ per cent per annum. Show how to compute $r$ and $n$.

As a practical example, a certain loan company will lend \$1200 to be repaid in 120 monthly installments of \$12.50 each. What is the rate of interest? Show how to compute it.

*Solution by J. E. Williams, Virginia Polytechnic Institute.*

Let $A$ be the original loan, $B$ the amount of each monthly installment, and $R$ the monthly rate of interest, $R>0$.

It is easy to see that the amount remaining after $n$ installments have been paid is

$$A(1+R)^n - BR^{-1}(1+R)^n + BR^{-1}. \tag{1}$$

If the debt is to be fully paid by $n$ installments, then (1) must be zero, and

$$(1+R)^n = 1/(1-AB^{-1}R). \tag{2}$$

In the practical example, $A=\$1200$, $B=\$12.50$ and $n=120$. We then have

$$(1+R)^{120} = 1/(1-96R) \quad \text{or} \quad 120\log(1+R) + \log(1-96R) = 0.$$

An approximate solution is $R=1/260$. Therefore the annual rate of interest is about 4.6%.

If $R$ is given, $n$ can be found by taking the logarithm of both sides of (2).

Also solved by C. F. Bowler, S. A. Corey, J. A. Duerksen, A. Pelletier, W. J. Patterson, A. W. Richardson, and F. L. Wilmer.

3307 [1928, 93]. *Proposed by C. N. Mills, Normal, Illinois.*

A pyramid whose faces are equilateral triangles and whose base is a square stands on a horizontal plane and faces the cardinal points. If $\alpha$ is the sun's altitude and $\beta$ the distance from the meridian and $\theta$ the vertical angle of the shadow prove that

$$\tan\theta = \tan 2\alpha \cos(\tfrac{1}{4}\pi - \beta).$$

I. *Solution by W. J. Patterson, The University of Western, Ontario.*

Let $ABCD$ be the square base of the pyramid with the edge $2S$ and center $O$, situated so that $DA$ is directed east and $AB$, north; also let $P$ be the vertex of the pyramid and $V$ the vertex of its shadow. Then

$$CO = OA = OP = 2^{1/2}S, \quad OV = 2^{1/2}S\cot\alpha, \quad \angle COV = \tfrac{1}{4}\pi + \beta, \quad \angle AVC = \theta.$$

Also

$$2S^2[\csc^2\alpha \pm 2\cot\alpha\cos(\tfrac{1}{4}\pi + \beta)]$$

is the value of $(VC)^2$ for the + sign and $(VA)^2$ for the − sign. The square of the area of $ACV$ is then

$$S^4 \sin^2\theta[\csc^4\alpha - 4\cot^2\alpha\cos^2(\tfrac{1}{4}\pi+\beta)].$$

But it is also $4S^4\cot^2\alpha\sin^2(\frac{1}{4}\pi+B)$; and after replacing $\sin^2\theta$ by $\tan^2\theta/(1+\tan^2\theta)$, we find

$$\tan\theta = 2\cot\alpha(\cot^2\alpha-1)^{-1}\sin(\tfrac{1}{4}\pi+\beta) = \tan 2\alpha\cos(\tfrac{1}{4}\pi-\beta).$$

In solving for $\tan\theta$ it should be noticed that for small values of $\alpha$ and $\beta$, $\tan\theta$ is positive.

II. *Solution by R. H. Sciobereti, Berkeley, California.*

In this solution the formula of the problem is obtained in a different way, and attention is directed to the fact that it is true only when the vertex $V$ of the shadow lies in certain parts of the plane. Formulae for other positions are derived. The results will be indicated in a condensed form using the above notation. Consider the two regions: I, the angle between the extensions of $CD$ and $AD$; II, the rectangular strip between the extension of $AD$, the parallel to it through $O$, and the part of $DC$ included between these parallels. Then $V$ lies in I when $2^{1/2}\cot\alpha\sin\beta>1$ and $2^{1/2}\cot\alpha\cos\beta>1$, and the formula of the problem is true. If $2^{1/2}\cot\alpha\cos\beta>1$ and $2^{1/2}\cot\alpha\sin\beta<1$, then $V$ lies in II and $\tan\theta=(2^{1/2}\cos\beta-\tan\alpha)/(\cot\alpha-2^{1/2}\cos\beta)$. If $2^{1/2}\cot\alpha\sin\beta>1$, $2^{1/2}\cot\alpha\cos\beta=1$, then $V$ lies on the extension of $CD$, and $\tan\theta=2/(\tan\beta-1)$. If $2^{1/2}\cot\alpha\sin\beta=1$, $2^{1/2}\cot\alpha\cos\beta>1$, then $V$ lies on the extension of $AD$, and $\tan\theta=2/(\cot\beta-1)$.

The remaining cases of the position of $V$ are easily obtained from these results.

Also solved by G. A. Yanosik.

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## NOTES AND NEWS

*Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.*

Professor G. H. Hardy, of Oxford University, who has been lecturing at Princeton University during the first semester of the present academic year on "Chapters in the theory of Functions," lectured at the Ohio State University on January 18, 1929 on the "Theory of Primes," and at the University of Chicago on January 21 on the "Analytical Theory of Numbers."

Professor H. Weyl, of Princeton University, lectured at the University of Iowa on December 2 and 3, 1928 on "Group Theory and Quantum Mechanics."

At the New York Meeting of the American Association for the Advancement of Science, Professor E. T. Bell, of the California Institute of Technology,

was elected vice-president and chairman of Section A, and Professor C. N. Moore, of the University of Cincinnati, was re-elected secretary; Professor H. L. Rietz, of the University of Iowa, was elected vice-president and chairman of Section K, and Professor C. F. Roos, of Cornell University, was elected secretary. At the same meeting Professor E. B. Wilson, professor of vital statistics, School of Public Health, Harvard University, was elected a member of the executive committee.

During the second half of the present acadamic year, beginning about February 1, Professor H. W. Tyler, of the Massachusetts Institute of Technology, will be on leave of absence in Washington, D. C., in order to establish there a permanent office of the American Association of University Professors of which he has acted as secretary for many years. Professor Tyler was re-elected secretary of this association at the recent New York Meeting.

A complete set of the Bulletin of the American Mathematical Society, including the three volumes of the New York Mathematical Society, is available for purchase. Also ten volumes of the Annals of Mathematics, 2nd. Series. Information may be had from the Secretary of the Association.

Mr. Paul K. Rees, of Texas Technological College, has been appointed assistant professor of mathematics at the University of Mississippi. Mr. Thomas A. Bickerstaff has been appointed to a teaching fellowship at the College.

Professor C. H. Richardson, formerly of Georgetown College and now professor of mathematics at Bucknell University, was recently elected by the Faculty of Bucknell University to serve as head of the department of mathematics for the current academic year.

Associate professor Robert Torrey of the University of Mississippi was granted leave of absence for the first semester of the present academic year. Professor Torrey was injured in a train wreck last June.

Professor Charles N. Wunder, head of the department of mathematics at the University of Mississippi, has been appointed the first dean of men at that university. Professor Wunder will continue as head of the department of mathematics.

The following graduate courses in mathematics are announced for the academic year 1929–30:

*The University of Chicago*—By Professor E. H. Moore: General analysis III, IV, V. By Professor H. E. Slaught: Differential equations; Advanced calculus; Definite integrals. By Professor L. E. Dickson: Theory of numbers; Advanced topics in algebra and the theory of numbers. By Professor G. A. Bliss: Applications of the calculus of variations I, II; Topics in the calculus of variations. By Professor A. C. Lunn: Statistical mechanics; Vector analysis; Dyadics and

crystal physics; Fourier series and Bessel functions; Units and dimensions; Vector analysis in Riemann-Einstein space. By Professor E. P. Lane: Analytic projective geometry; Metric differential geometry; Solid analytic geometry. By Professor W. D. MacMillan: Theoretical mechanics I, II, III; The problem of three bodies; Dynamics of rigid bodies. By Professor M. I. Logsdon: Theory of functions of a complex variable; Algebraic geometry I, II. By Professor L. M. Graves: Vectors and matrices in higher algebra; Theory of functions of a real variable; Integral and functional equations. By Professor R. W. Barnard: General analysis I, II. By Professor W. Bartky: Stellar statistics. By an instructor to be appointed: Theory of equations; Limits and series; Algebraic invariants. Thesis work and directed reading and research are offered in the foundations of mathematics and general analysis by Professors Moore and Barnard; in algebra and the theory of numbers by Professors Dickson and Barnard; in analysis by Professors Bliss and Graves; in applied mathematics by Professors Lunn, MacMillan, and Bartky; in differential geometry by Professor Lane; and in algebraic geometry by Professor Logsdon.

The following courses in mathematics are announced for the summer of 1929:

*The University of Chicago*, first term, June 17–July 24; second term, July 25–August 30. In addition to the usual courses in plane analytic geometry and differential calculus, the following advanced courses will be offered. By Professor H. E. Slaught: Differential equations; Elliptic integrals. By Professor A. C. Lunn: Relativity; Lattices and crystal groups. By Professor A. B. Coble (The University of Illinois): Analytic projective geometry; Algebraic geometry. By Professor L. M. Graves: Introduction to analysis; Functions of lines. By Professor R. W. Barnard: Theory of equations; Metric differential geometry. By Professor W. Bartky: Theoretical mechanics; Modern theories of analytic differential equations. By Professor C. C. MacDuffee (Ohio State University): Theory of numbers; Theory of algebraic numbers; Linear algebras. By Professor H. S. Everett: Advanced calculus.

Professor H. B. Fine, of Princeton University, was fatally injured by an automobile on the evening of Friday, December 21 and died about one A.M. on December 22, 1928. He was seventy years of age.

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## SUGGESTIONS TO AUTHORS

**Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.**

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write $\epsilon$, not $\varepsilon$. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: 1, 2, $\cdots$, $n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, l means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: $l$. Similarly, O means "zero"; to indicate "cap O," backspace and overprint period; thus: $O$.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

---

Light Face Greek—$\alpha\ \beta\ \gamma\ \cdots$ (all) A B $\Gamma\ \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma\ \cdots}$ (all except o).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma\ \ldots}$ (all except o).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \gamma\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \kappa\ \lambda\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ e\ f\ p\ q}$ $\mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ K\ L\ M\ P\ Q\ R}$ $\mathfrak{S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b\ A\ B\ D}$.

Script (special font) $\mathcal{A\ B\ C}\ \cdots$ (all). No lower case manufactured.

* Hebrew— ב כ א ד troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{i}\ \overline{j}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{o}\ \overline{P}\ \overline{p}$ $\overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \bar{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted $\gamma$ readily available).

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* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

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Witness: ..... Signature .....

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[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

# Missing Numbers of the Monthly

Cash, or credit toward future dues, will be given for certain single numbers as follows, up to a limited number of copies: February, March, May, September, 1913; September, 1914; February, March, April, June, 1915; February, September, 1918—fifty cents; September, 1915—seventy-five cents; May, 1915—one dollar. (See MONTHLY, March, 1921, p. 152); October, 1920; August-September, October, 1921; May, September, October, 1924; May, June-July, November, 1926—forty-five cents.

**Address all communications to the**

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## DIRECTORY

**EDITORIAL CORRESPONDENCE** should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

**BOOKS FOR REVIEW** should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

**BUSINESS CORRESPONDENCE** should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

### MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.

INDIANA, Culver Military Academy, May 3-4.

IOWA.

KANSAS, Topeka, Kansas, February 2.

KENTUCKY.

LOUISIANA-MISSISSIPPI.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.

MICHIGAN, Ann Arbor, Mich., March 16.

MINNESOTA, St. Paul, May 8.

MISSOURI, Kansas City, Mo., November.

NEBRASKA.

OHIO, Columbus, Ohio, April 4.

PHILADELPHIA, University of Pennsylvania, November 30.

ROCKY MOUNTAIN, Greeley, Colo., April 12-13.

SOUTHEASTERN, Macon, Ga., April 29.

SOUTHERN CALIFORNIA, University of Redlands, March 9.

TEXAS, Houston, Texas, June 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

## THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

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## THE THIRTEENTH ANNUAL MEETING OF THE ASSOCIATION

The thirteenth annual meeting of the Mathematical Association of America was held at Columbia University, New York, New York on Friday and Saturday, December 28-29, 1928 in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. Three hundred forty were in attendance at these meetings, among them the following two hundred twenty-one members of the Association.

C. R. Adams, Brown University
Beatrice Aitchison, Metropolitan Life Ins. Co., New York, N. Y.
Joseph Allen, College of the City of New York
R. C. Archibald, Brown University
C. L. Arnold, Ohio State University
H. E. Arnold, Wesleyan University
T. B. Ashcraft, Colby College
C. S. Atchison, Washington and Jefferson College

Clara L. Bacon, Goucher College
H. C. Barber, Phillips Exeter Academy
I. A. Barnett, University of Cincinnati
Helen Barton, North Carolina College for Women
R. D. Beetle, Dartmouth College
A. A. Bennett, Brown University
Theodore Bennett, Princeton University
C. A. Bergstresser, Jamaica High School, New York, New York
W. J. Berry, Brooklyn Polytechnic Institute
O. F. H. Bert, Washington and Jefferson College
G. D. Birkhoff, Harvard University
L. M. Blumenthal, Johns Hopkins University
Joseph Bowden, Adelphi College
J. W. Bradshaw, University of Michigan
R. S. Burington, Case School of Applied Science
F. J. Burkett, Trinity College

W. D. Cairns, Oberlin College
Minnie W. Caldwell, Chowan College
B. H. Camp, Wesleyan University
A. D. Campbell, Syracuse University
Mildred E. Carlen, Brown University
R. D. Carmichael, University of Illinois
Evelyn T. Carroll, Wells College